5. The Intermediate Value Theorem (23/9/2021)

Goals:

(1) The IVT
   (a) With given endpoints
   (b) Free-form (you find endpoints)
(2) (if there's time) The derivative

Last Time.

Continuity: f is cts at a if

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

("no break in graph")

Promise: If f is defined by formula near a, f is cts at a.

Two ideas: (1) check continuity by computing limits
(2) Use continuity to evaluate limits

Theorem: If f is cts on $[a,b]$ then f takes every value between $f(a)$, $f(b)$.

("no jumps")

Use IVT: want to know that $f(x) = A$ without knowing which x makes it true.

E.g.: To show $f(x) = A$ has a solution, if A is between $f(a)$, $f(b)$, IVT says there is one.

Two difficulties:

(1) conceptual: "solve" an equation without find the solution

(2) technical: often have to use inequalities

Worksheet (1)

Person A ~~w~~ starts at bottom of hill,
Person B " " top " ".
They start on the path. Show that they meet!

let $f(t) =$ ~~posi~~ height of position of person A | "giving names"
$g(t) =$ " " " " " B

want time t s.t. $f(t) = g(t) \iff f(t) - g(t) = 0$

~~slow~~ If f, g cts so is $f-g$. In morning $(f-g)(0) = f(0) - g(0)$
$= -H$
H = height of hill

Math 100 – WORKSHEET 5
THE IVT

1. The Intermediate Value Theorem

(1) Show that $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.

Since f is defined by formula, it's cts on $[0,1]$.

$f(0) = 1$, $f(1) = -2$, and $-2 < 0 < 1$.

By the IVT, there is c, $0 < c < 1$, s.t. $f(c) = 0$

H = height of the hill

Want: $f(c) - g(c) = 0$ $\quad$ (c = time of meeting)

$$\left.\begin{aligned} f(0) - g(0) &= -H \\ f(\text{evening}) - g(\text{evening}) &= H \end{aligned}\right\} \quad -H < 0 < H$$

By IVT there is c s.t. $f(c) - g(c) = 0$

and thus $f(c) = g(c)$

(to solve $f(t) = g(t)$, consider $(f-g)(t) = 0$ instead)

Worksheet (2)

(2) (Final 2011) Let $y = f(x)$ be continuous with domain $[0,1]$ and range in $[3,5]$. Show the line $y = 2x + 3$ intersects the graph of $y = f(x)$ at least once.

The line intersects the graph at x if

$$f(x) = 2x + 3$$ ← expressed problem as an equation

↙ subtracted

let $g(x) \stackrel{\text{def}}{=} f(x) - 2x - 3$ (want $0 \le c \le 1$ s.t. $g(c) = 0$)

~~[illegible]~~ f is cts by hypothesis, $2x+3$ is cts (polynomial)

so $g(x) = f(x) - (2x+3)$ is also cts.

↑ checked continuity

$g(0) = f(0) - 3 \geq 0$ $\quad$ $g(1) = f(1) - 5 \in [-2, 0]$

so $g(1) \le 0$.

$f(0) \geq 3$

↑ evaluated at two pts

By the IVT, there is $0 \le c \le 1$ s.t. $g(c) = 0$

↖ invoked IVT

i.e. $f(c) - (2c+3) = 0$

i.e. $f(c) = 2c + 3$

endgame: Converted info $g(c)=0$ to a solution of problem

(3) $\sin x = x + 1$ has a solution.

Let $f(x) = \sin x - (x+1)$. Since f is defined by formula it's cts everywhere

$f(0) = -1 < 0$, $f(-\pi) = \sin(-\pi) - (-\pi + 1)$
$= \pi - 1 > 0$

$f(\pi) = \sin \pi - (\pi + 1)$
$= -(\pi+1) < 0$

By the IVT there is c ~~st~~ $-\pi < c < 0$ s.t. $f(c) = 0$

i.e. $\sin c = c + 1$.

Alternative:

$f(1000) = \sin 1000 - 1000 - 1 \leq 1 - 1000 - 1 = -1000 < 0$

$f(-1000) = -\sin 1000 - (-999) = 999 - \sin 1000 \geq 998 > 0$

$\sin 1000 \leq 1$

(4) (Final 2015) Show that the equation $2x^2 - 3 + \sin x + \cos x = 0$ has at least two solutions.

let $f(x) = 2x^2 - 3 + \sin x + \cos x$.

f is cts (defined by formula)

$f(10) = 200 - 3 + \sin 10 + \cos 0 \geq 200 - 3 - 1 - 1 \geq 195 > 0$

$f(-10) = 200 - 3 + \sin(-10) + \cos(-10) \geq 200 - 3 - 1 - 1 \geq 195 > 0$

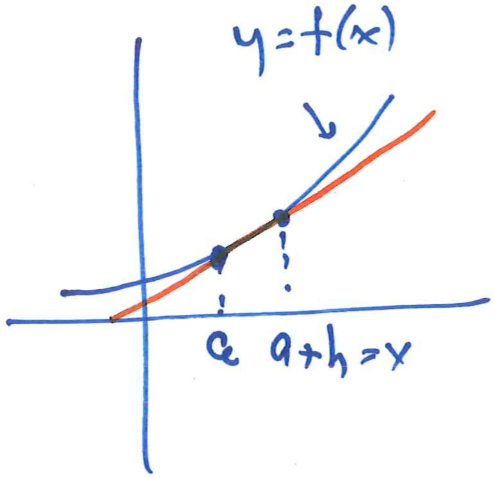
$f(0) = -3 + 0 + 1 = -2 < 0$

By IVT. f has a zero between $(-10, 0)$

and also a zero between $(0, 10)$.

The Derivative

Recall from lecture 1:



To find slope of the line tangent to graph of $y=f(x)$ at point a,
use a limiting process:

① use pt x near a to draw line through $(a, f(a))$
$(x, f(x))$

it has slope $\dfrac{f(x)-f(a)}{x-a}$.

② let $x \to a$ then if $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists,
it's the slope of the tangent line.

Def: Call $\displaystyle\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ the derivative of f at a.

We write $f'(a)$ for this number.

If limit exists we say f is differentiable at a

Question: we already know that derivative of x^2 is $2x$
What's the point of this definition?