4. Continuity; The Intermediate Value Theorem (21/9/2021)

Goals.

(1) Continuity
  (a) Definition
  (b) Gluing of functions
(2) The Intermediate Value Theorem
  (a) With given endpoints
  (b) Free-form
~~(3) The derivative~~
  ~~(a) Definition~~
  ~~(b) Some calculations~~

Last Time.

(1) ∞ limits:

Prototypical example:

$$f(x) = \frac{g(x)}{h(x)}$$

if $g(x) \xrightarrow[x\to a]{} G \neq 0$, $h(x) \xrightarrow[x\to a]{} 0$ then f blows up at a.

examine signs of $g(x)$, $h(x)$ to see if the limit is $+\infty$ or $-\infty$ in the extended sense.

Today: TA office hours
13:30-15:30
At ORCH 3061

Instructor:
21:30-23:00
on Zoom

Example 6: $\lim\limits_{x\to 0} \dfrac{x-5}{x^2(x+3)} = \lim\limits_{x\to 0} \dfrac{1}{x^2}\cdot\left(\dfrac{x-5}{x+3}\right)$

As $x\to 0$, $\cancel{\dfrac{1}{x^2}\to\infty}$, $\dfrac{x-5}{x+3} \xrightarrow[x\to 0]{} \dfrac{0-5}{0+3} = -\dfrac{5}{3} < 0$

$x^2 \to 0$

causes blowup

near $x=0$, $\dfrac{1}{x^2} > 0$, $\dfrac{x-5}{x+3} < 0$ (has negative limit)

So $\lim\limits_{x\to 0} \dfrac{x-5}{x^2(x+3)} = -\infty$

Continuity

Idea: Graph of f has no "breaks"

Def: Let f be defined near a (including a)
Say that f is continuous at a if:

(1) $\lim\limits_{x\to a} f(x)$ exists; (2) $\lim\limits_{x\to a} f(x) = f(a)$

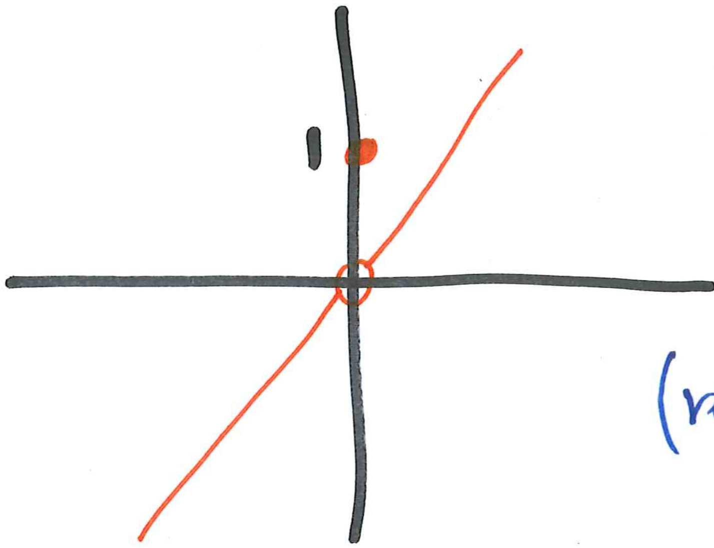
$\iff$

$$\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$$

Recall: $\lim\limits_{x\to a} f(x)$ does not "see" $f(a)$

Example: $f(x) = \begin{cases} x & x \neq 0 \\ 1 & x = 0 \end{cases}$

is discontinuous at $x=0$



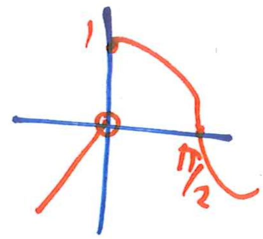
here: $\lim_{x \to 0} f(x) = \lim_{x \to 0} x = 0 \neq f(0)$

(recall: if $\lim_{x \to a} f(x) = L$ then $f(a)$ can be anything)

But $g(x) = x$ is continuous at $x=0$

Promise: if f is given by an expression, it is continuous ~~when~~ wherever the expression makes sense

Worksheet (1)



Math 100 – WORKSHEET 4
CONTINUITY; THE IVT

1. Continuity

(1) Which of these functions are continuous everywhere? Why?

(a) $f(x) = \begin{cases} x & x < 0 \\ \cos x & x \geq 0 \end{cases}$

On $(-\infty,0)$, $(0,\infty)$ f is given by a formula, so continuous. Only need to check $x=0$. At $x=0$,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x = 0$; $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \cos x = \cos 0 = 1 = f(0)$

so $\lim_{x\to 0} f(x) \neq f(0)$, so f is discontinuous at 0.

(b) $g(x) = \begin{cases} x & x < 0 \\ \sin x & x \geq 0 \end{cases}$

On $(-\infty,0)$, $(0,\infty)$ g is given by formula, hence continuous. At $x=0$:

$\lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} x = 0$; $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} \sin x = \sin 0 = 0$

so g is cts at 0, hence everywhere

Date: 21/9/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Continuous

(2) Let $f(x) = \frac{x^3 - x^2}{x-1}$.

(a) Why is $f(x)$ discontinuous at $x = 1$? f is not defined there

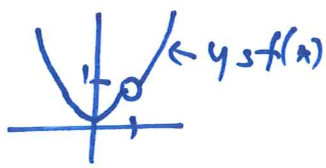
(b) Find b such that $g(x) = \begin{cases} f(x) & x \neq 1 \\ b & x = 1 \end{cases}$ is continouous everywhere.

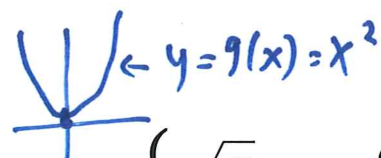
To make g cts at 1, need $\lim_{x \to 1} g(x) = g(1) = b$

Now $\lim_{x \to 1} g(x) = \lim_{x \to 1} f(x) = \lim_{x \to 1} \frac{x^3 - x^2}{x-1} = \lim_{x \to 1} \frac{x^2(x-1)}{x-1} =$

$= \lim_{x \to 1} x^2 = 1$, so $b = 1$ works

In a picture:





(c) Find c, d such that $h(x) = \begin{cases} \sqrt{x} & 0 \leq x < 1 \\ c & x = 1 \\ d - x^2 & x > 1 \end{cases}$

is continuous.

h is given by formula on $[0,1)$, $(1, \infty)$ so cts there.

At $x=1$: $\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} \sqrt{x} = \sqrt{1} = 1$ (previous knowledge)

$h(x) = \sqrt{x}$ if $x < 1$

$\lim_{x \to 1^+} h(x) = \lim_{x \to 1^+} (d - x^2) = d - 1$.

$h(1) = c$. So h is cts if $1 = c = d - 1$

i.e. if $\boxed{c = 1,\ d = 2.}$

Worksheet 5 (2) (a), (b):

$$g(x) = \begin{cases} x^2 & x \neq 1 \\ b & x = 1 \end{cases}$$

$\lim_{x \to 1} g(x) = 1$ no matter what b is.

$= \lim_{x \to 1} x^2 = 1^2 = 1$

If $b = 1$, $\lim_{x \to 1} g(x) = g(1)$ $(g(1) = b)$

if $b \neq 1$, $\lim_{x \to 1} g(x) \neq g(1)$

(d) (Final 2013) For which value of the constant c is
$$f(x) = \begin{cases} cx^2+3 & x \geq 1 \\ 2x^3 - c & x < 1 \end{cases} \text{ continuous on } (-\infty, \infty)?$$

On $(-\infty, 1)$, $(1, \infty)$ f is given by formula, hence cts

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2x^3 - c) = 2\cdot 1^3 - c = 2 - c$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (cx^2+3) = c\cdot 1^2 + 3 = c+3 = f(1)$$

So f is cts at 1 if $2 - c = c + 3$ ie. $\boxed{c = -\frac{1}{2}.}$

(3) Where are the following functions continuous?

$f(x) = \frac{1}{\sqrt{7-x^2}}$ where $7 - x^2 > 0$ ie. on $(-\sqrt{7}, +\sqrt{7})$

$g(x) = \frac{x^2+2x+1}{2+\cos x}$ on $\mathbb{R}$ $\left(2+\cos x \geq 2-1 > 0\right.$ for all x $\left.\right)$

$h(x) = \frac{2+\cos x}{x^2+2x+1} = \frac{2+\cos x}{(x+1)^2}$ cts on $(-\infty, -1) \cup (-1, \infty)$

$k(x) = \log(\sin x)$ cts if $\sin x > 0$,

ie. $\bigcup_{k\in\mathbb{Z}} (0 + 2\pi k, \pi + 2\pi k) = \bigcup_{k\in\mathbb{Z}} (2\pi k, 2\pi(k+\frac{1}{2}))$

(4) (Final 2011) Suppose f, g are continuous such that $g(3) = 2$ and $\lim_{x\to 3}(xf(x) + g(x)) = 1$. Find $f(3)$.

$1 = \lim_{x\to 3}(xf(x) + g(x)) = (\lim_{x\to 3} x)(\lim_{x\to 3} f(x)) + \lim_{x\to 3} g(x)$

limit laws

$= 3 \cdot f(3) + g(3) = 3f(3) + 2$ so $\boxed{f(3) = -\frac{1}{3}}$

f, g are cts at 3

2. The Intermediate Value Theorem

Theorem. *Let $f(x)$ be continuous for $a \le x \le b$. Then $f(x)$ takes every value between $f(a), f(b)$.*

(5) Show that $f(x) = 2x^3 - 5x + 1$ has a zero in $0 \le x \le 1$.