19. Curve Sketching (18/11/2021)

Goals.

(1) Curve sketching protocol
(2) Examples from past exams

Last Time. Effects of MVT on graph

(1) $f'>0 \Rightarrow f$ increasing ↗
$f'<0 \Rightarrow f$ decreasing ↘

(2) $f''>0 \Rightarrow f$ concave up ∪
$f''<0 \Rightarrow f$ concave down ∩
(change of concavity at inflection points)

either can be used to detect local extrema

(0) Can also look at values of f: domain, where $f>0$, $f<0$, vertical/horizontal asymptotes

Example: $f(x) = x^{2/3}(x-1)$

f is defined on $\mathbb{R} = (-\infty, \infty)$ $f(x)=0$ at $x=0,1$.

f is negative on $(-\infty,0)$, negative on $(0,1)$, positive on $(1,\infty)$

$x^{2/3} = (x^{1/3})^2 \geq 0$ for all x, so $f(x)$ has same sign as $x-1$ (if $x\neq 0$)

Can touch axis without changing sign

[As $|x|\to\infty$, $|f(x)| \sim |x|^{5/3}$] no horizontal asymptotes. $\lim_{x\to\pm\infty} f(x) = \pm\infty$

$f(x) \sim x^{5/3}$

$$f'(x) = \frac{2}{3}x^{-1/3}(x-1) + x^{2/3} = \frac{2(x-1)+3x}{3x^{1/3}} = \frac{5x-2}{3x^{1/3}}$$

$$= \left(x^{5/3} - x^{2/3}\right)' = \frac{5}{3}x^{2/3} - \frac{2}{3}x^{-1/3} = \frac{5x-2}{3x^{1/3}}$$

$\Rightarrow f'$ has a singular pt at $x=0$

critical pt at $x = 2/5$

$\Rightarrow f'$ positive on $(-\infty, 0)$, negative on $(0, \frac{2}{5})$, positive on $(\frac{2}{5}, \infty)$

$\begin{pmatrix} 5x-2<0 \\ x^{1/3}<0 \end{pmatrix}$ $\begin{pmatrix} 5x-2<0 \\ x^{1/3}>0 \end{pmatrix}$ $\begin{pmatrix} 5x-2>0 \\ x^{1/3}>0 \end{pmatrix}$

$\Rightarrow f$ increases on $(-\infty, 0)$, has a local max at $x=0$, decreases on $(0, \frac{2}{5})$, has a local min at $x = 2/5$, increases on $(\frac{2}{5}, \infty)$.

$$f''(x) = \frac{5}{3}\cdot\frac{2}{3}\cdot x^{-1/3} - \frac{2}{3}\cdot\left(-\frac{1}{3}\right)\cdot x^{-4/3} = \frac{10x+2}{9x^{4/3}}$$

So $f''(x)$ undef at $x=0$, $f(-\frac{1}{5}) = 0$, $x^{4/3} > 0$ if non-zero

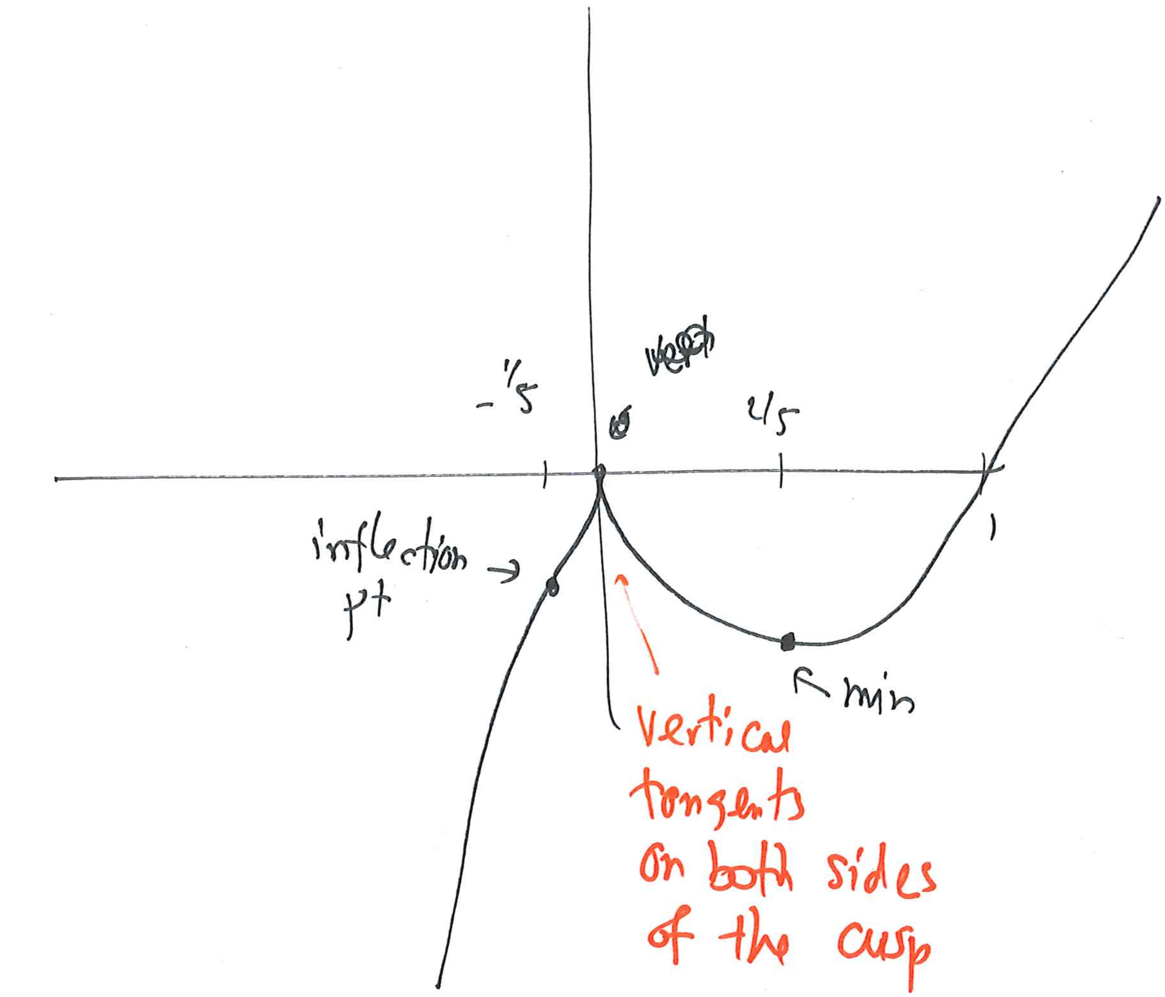
So $f''(x)$ has same sign as $10x+2$.

So $f''(x) < 0$ on $(-\infty, -\frac{1}{5})$ (f concave down)

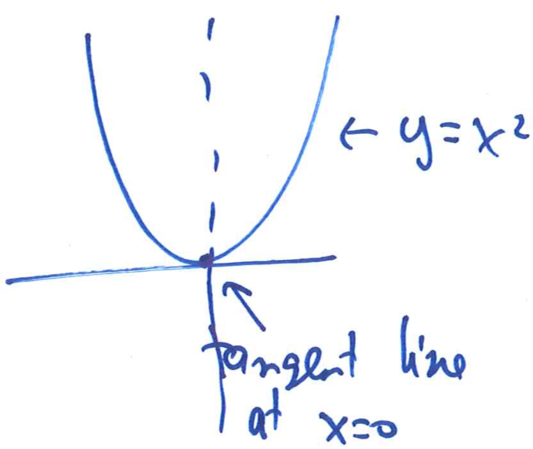
f has an inflection point at $x = -\frac{1}{5}$

$f''(x) > 0$ on $(\frac{1}{5}, 0)$ (f concave up)

$f''(x) > 0$ on $(0, \infty)$ (" " ")



$-1/5$
0
$4/5$
1
inflection pt
min
vertical tangents on both sides of the cusp



$\leftarrow y = x^2$
tangent line at $x=0$

[16] **4.** Let $f(x) = x\sqrt{3-x}$.

(a) (2 marks) Find the domain of $f(x)$.

f is defined where $3-x \geq 0$
i.e. if $x \leq 3$

Answer: $x \leq 3$ or $(-\infty, 3]$

(b) (4 marks) Determine the x-coordinates of the local maxima and minima (if any) and intervals where $f(x)$ is increasing or decreasing.

$$f'(x) = \sqrt{3-x} + x \cdot \frac{1}{2\sqrt{3-x}} \cdot (-1) = \frac{2(3-x) - x}{2\sqrt{3-x}} = \frac{6-3x}{2\sqrt{3-x}} = \frac{3}{2} \cdot \frac{2-x}{\sqrt{3-x}}$$

so critical pt at $x=2$, vertical tangent at $x=3$

$f' > 0$ on $(-\infty, 2)$, $f' < 0$ on $(2,3)$

so f is increasing on $(-\infty, 2)$,
has a local maximum at $x=2$
f is decreasing on $(2,3)$

(c) (2 marks) Determine intervals where $f(x)$ is concave upwards or downwards, and the x-coordinates of inflection points (if any). You may use, without verifying it, the formula $f''(x) = (3x-12)(3-x)^{-3/2}/4$.

$$f''(x) = \frac{3(x-4)}{4(3-x)^{3/2}}$$

undefined at 3, negative otherwise
$(x \leq 3 \Rightarrow x - 4 < 0)$

so f is concave down.

Question 4 continued on the next page...

Name:

Question 4 continued

(d) (2 marks) There is a point at which the tangent line to the curve $y = f(x)$ is vertical. Find this point.

Answer $x = 3$

$$\lim_{x\to 3^-} f'(x) = \lim_{x\to 3^-} \frac{3}{2}\frac{2-x}{\sqrt{3-x}} = -\infty$$

(e) (2 marks) The graph of $y = f(x)$ has no asymptotes. However, there is a real number a for which $\lim\limits_{x\to-\infty} \dfrac{f(x)}{|x|^a} = -1$. Find the value of a.

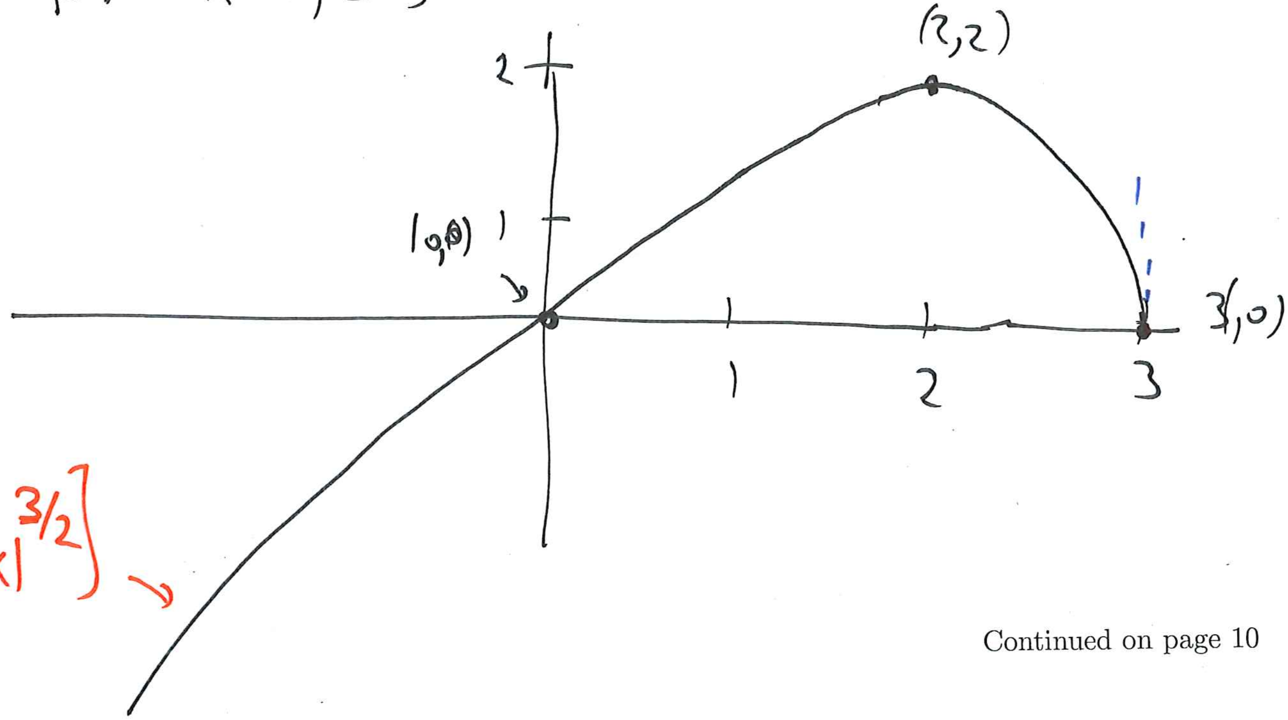
Answer $a = 3/2$

as $x \to -\infty$, $3 - x \sim |x|$ so
$f(x) \sim -|x| \cdot \sqrt{|x|} \sim -|x|^{3/2}$

$$\lim_{x\to-\infty} \frac{f(x)}{|x|^{3/2}} = \lim_{x\to-\infty} \frac{x}{|x|}\cdot\frac{\sqrt{3-x}}{|x|^{1/2}} = (-1)\lim_{x\to-\infty}\sqrt{\frac{-x}{|x|}+\frac{3}{|x|}} = -\lim_{x\to\infty}\sqrt{1+\frac{3}{x}} = -1$$

(f) (4 marks) Sketch the graph of $y = f(x)$, showing the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above and also all x-intercepts.

$f(x) = 0$ for $x = 0, 3$, $f(2) = 2\sqrt{3-2} = 2$



$[\sim -|x|^{3/2}]$

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[14] **4.** Let

$$f(x) = \begin{cases} \frac{4}{\pi}\tan^{-1} x, & \text{if } x \geq 1, \\ 2 - x^4, & \text{if } x < 1. \end{cases}$$

[*Note:* Another notation for $\tan^{-1}$ is arctan.]

(a) (3 marks) Show that $f(x)$ is continuous at $x = 1$.

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{4}{\pi}\arctan x = \frac{4}{\pi}\arctan 1 = f(1) = \frac{4}{\pi}\cdot\frac{\pi}{4} = 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (2-x^4) = 2-1^4 = 1 = f(1)$ ✓

(b) (1 mark) Determine the equations of any asymptotes (horizontal, vertical or ~~slant~~).

f cts everywhere, so no vertical asymptotes

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4}{\pi}\arctan x = \frac{4}{\pi}\cdot\frac{\pi}{2} = 2$ So $y=2$ is a horizontal asymptote as $x\to\infty$

as $x\to-\infty$ $f(x) \sim -x^4$ so no horizontal asymptote there (or $\lim_{x\to-\infty} f(x) = -\infty$

(c) (4 marks) Determine all critical numbers, open intervals where f is increasing or decreasing, and the x-coordinates of all local maxima or local minima (if any).

$$f'(x) = \begin{cases} \frac{4}{\pi(1+x^2)} & x>1 \\ DNE & x=1 \\ -4x^3 & x<1 \end{cases}$$

$\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} = \frac{4}{\pi(1+1^2)}$ $2/\pi$

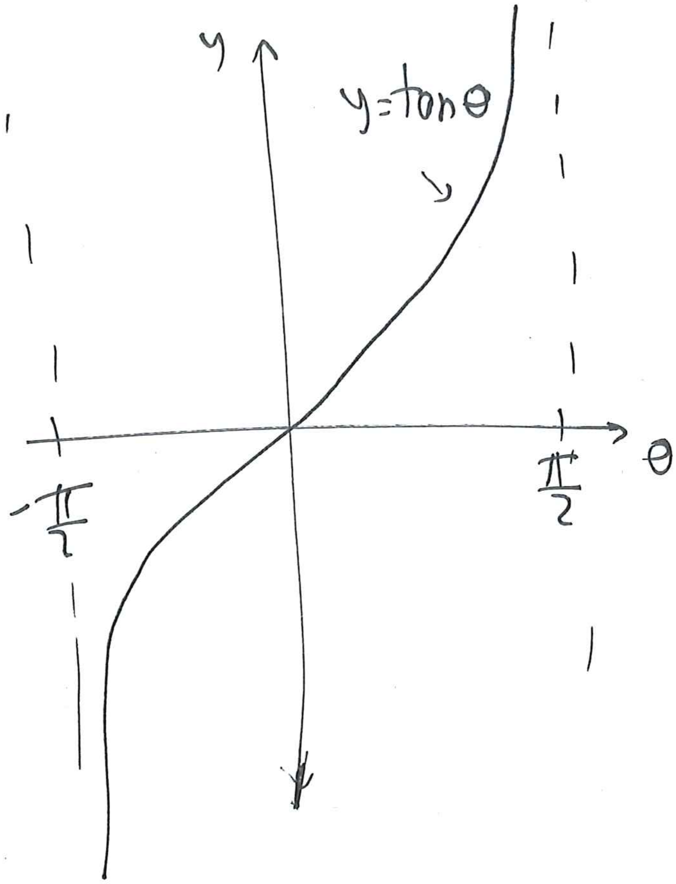
$\lim_{h\to 0^-} \frac{f(1+h)-f(1)}{h} = -4\cdot 1^3 = -4$

f has a singular point at $x=1$, critical pt at $x=0$

$f'(x) > 0$ on $(-\infty, 0)$, $f' < 0$ on $(0,1)$, $f' > 0$ on $(1,\infty)$

So f is increasing on $(-\infty,0)$, has a local max at $x=0$,
f is decreasing on $(0,1)$, has a local min at $x=1$,
f is increasing on $(1,\infty)$

Question 4 continues on the next page...



$$\lim_{y\to\infty} \arctan y = \frac{\pi}{2} \quad \left(\lim_{\theta\to\frac{\pi}{2}^{+}} \tan\theta = \infty\right)$$

$$\lim_{y\to-\infty} \arctan y = -\frac{\pi}{2} \quad \left(\lim_{\theta\to-\frac{\pi}{2}^{-}} \tan\theta = -\infty\right)$$

Question 4 continued

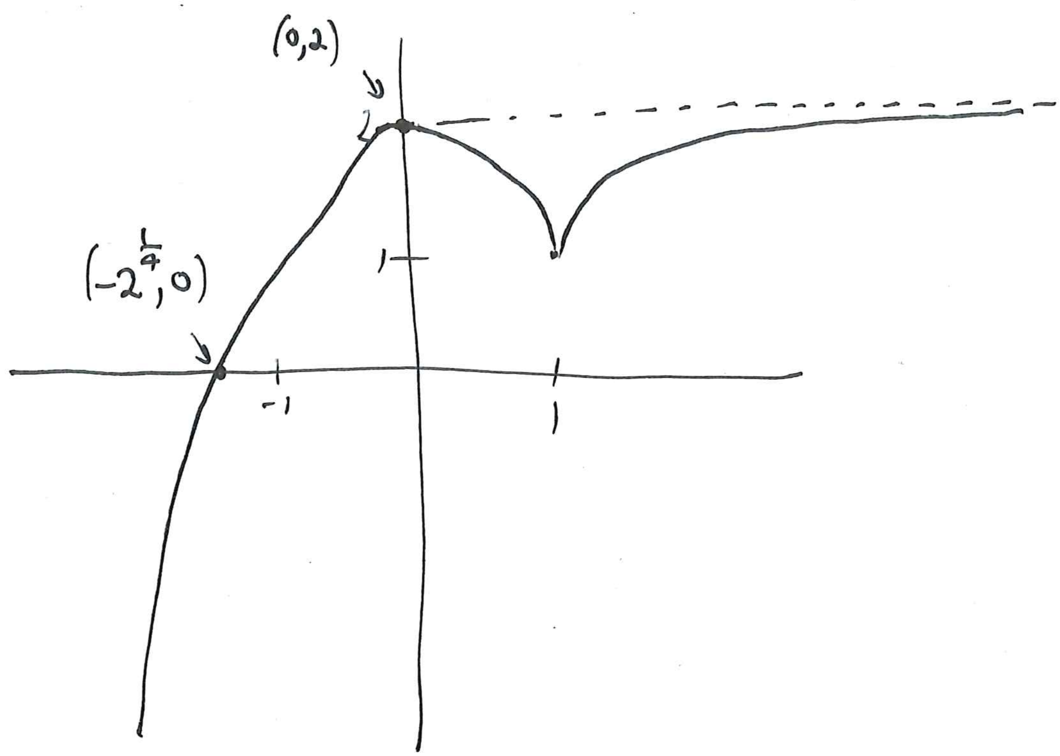
(d) (2 marks) Determine open intervals where the graph of f is concave upwards or concave downwards, and the x-coordinates of all inflection points (if any).

We have $f''(x) = \begin{cases} -\frac{4}{\pi}\frac{1}{(1+x^2)^2}\cdot 2x = -\frac{8x}{\pi(1+x^2)^2} & x>1 \\ -12x^2 & x<1 \end{cases}$

Both expressions are negative $(8x>0$ if $x>1)$ except $f''(0)=0$.

f is therefore concave down on $(-\infty,1)$ and on $(1,\infty)$, and has no inflection points

(e) (4 marks) Sketch the curve $y = f(x)$, showing all the features given in items (a) to (d) above and giving the (x, y) coordinates for all points occurring above (if any).



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