11. Inverse Trig; Related Rates (19/10/2021)

Goals.

(1) Evaluating inverse trig functions
(2) Differentiating inverse trig functions
(3) Related Rates

Last Time.

$(\log f)' = \frac{f'}{f} \Rightarrow f' = f \cdot (\log f)'$

(helps if f is a product or power)

Implicit diff: if we have a relation between x,y we can diff wrt x and solve for $\frac{dy}{dx}$ as a function of (x,y) on the curve.

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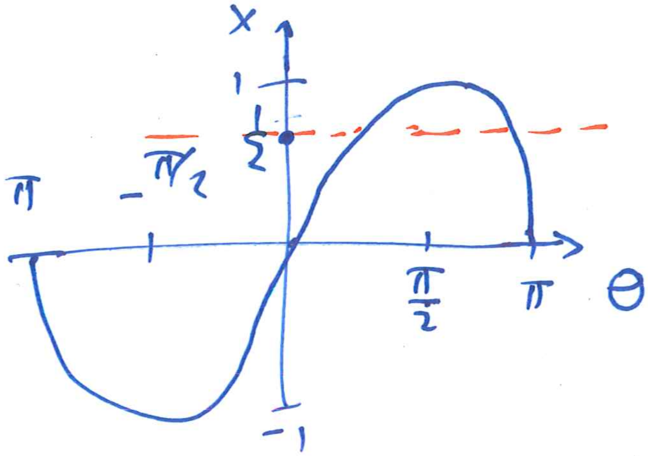
e.g. if $y^x = x^y$ in x^y both x,y are variable:

not x^a or a^y. Can't differentiate directly.

So take log of both sides, get new relation ..

Arcsine, Arctangent, etc

Q: does $\sin\theta$, $\theta \in \mathbb{R}$ have an inverse function?



no: $\sin\theta$ takes same value many times

$\Rightarrow$ restrict domain. Range of $\sin\theta$: $[-1, 1]$

attained on $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. There we have an inverse

Def: $\theta = \arcsin x$ if $\sin\theta = x$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(some people write $\sin^{-1} x$, but beware $\sin^{-1} x \neq \frac{1}{\sin x}$)

Similarly $\theta = \arccos x$ if $\cos\theta = x$ and $\theta \in [0, \pi]$

(both only defined if $-1 \le x \le 1$)

Finally: $\theta = \arctan x$ if $\tan\theta = x$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

Math 100 – WORKSHEET 11
INVERSE TRIG FUNCTIONS; RELATED RATES

1. Inverse trig functions

(1) Evaluation

(a) (Final 2014) Evaluate $\arcsin\left(-\frac{1}{2}\right)$; Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$

$\sin\frac{\pi}{6} = \frac{1}{2}$ so $\sin\left(-\frac{\pi}{6}\right) = -\frac{1}{2}$ so $\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$.

$\frac{31}{11}\pi = 3\pi - \frac{2}{11}\pi \notin \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ so $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right) \neq \frac{31\pi}{11}$.

$\left(\sqrt{(-5)^2} \neq -5\right)$ how to find the angle? think of $\underline{\sin\theta}$.

$\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31}{11}\pi - 2\pi\right) = \sin\left(\frac{9}{11}\pi\right) = \sin\left(\pi - \frac{9}{11}\pi\right) = \sin\left(\frac{2}{11}\pi\right)$

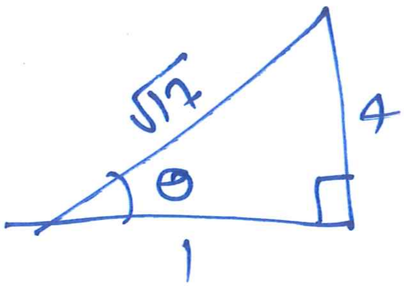
$\sin\theta = \sin(\pi - \theta)$

(c.f. $\cos(\pi - \theta) = -\cos\theta$)

$$\boxed{\text{so } \arcsin\left(\sin\frac{31}{11}\pi\right) = \frac{2}{11}\pi}$$

Date: 19/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) (Final 2015) Simplify $\sin(\arctan 4)$

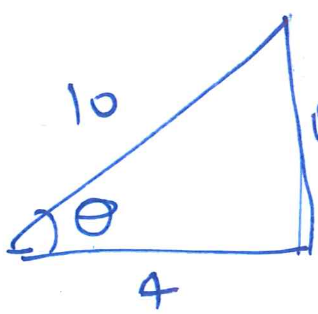


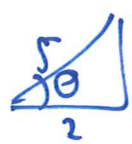
want $\tan\theta = 4$

choose sides 4, 1

so $\sin\theta = \sin(\arctan 4) = \frac{4}{\sqrt{17}}$

(c) Find $\tan(\arccos(0.4))$



(or 

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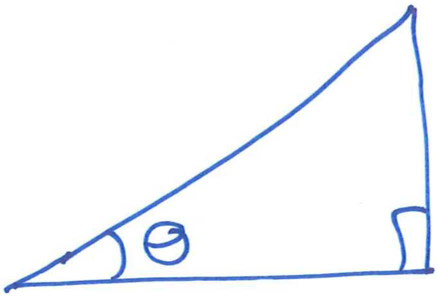
so $\tan(\arccos(0.4)) = \frac{\sqrt{84}}{4} = \frac{\sqrt{21}}{2}$.

What about $\sin(\arccos x)$ or $\tan(\arcsin x)$ . . ?

or ~~arctan(cos x)~~



① draw triangle, put θ

② choose sides to make θ what it need to be

③ use Pythagoras to find missing side

④ evaluate

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derivatives: let $\theta = \arcsin x$ then $\sin\theta = x$

so $(\cos\theta)\cdot\frac{d\theta}{dx} = \frac{d}{dx}(\sin\theta) = \frac{d}{dx}x = 1$

so $\frac{d\theta}{dx} = \frac{1}{\cos\theta} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\sin^2\theta}} = \frac{1}{\sqrt{1-x^2}}$

memorize: $\boxed{\frac{d(\arcsin x)}{dx} = \frac{1}{\sqrt{1-x^2}},\quad \frac{d(\arctan x)}{dx} = \frac{1}{1+x^2}}$

(because $\cos\theta = \sin(\frac{\pi}{2}-\theta)$, $\arccos x = \frac{\pi}{2} - \arcsin x$

so $\boxed{\frac{d}{dx}(\arccos x) = -\frac{d}{dx}(\arcsin x)}$

(2) Differentiation

(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$ $= \frac{1}{\sqrt{1-(2x)^2}} \cdot 2$ $\frac{d(2x)}{dx}$

(chain rule)

(b) Find the line tangent to $y = \sqrt{1 + (\arctan(x))^2}$ at the point where $x = 1$.

$\frac{dy}{dx} = \frac{1}{2\sqrt{1+(\arctan x)^2}} \cdot 2(\arctan x) \cdot \frac{1}{1+x^2}$

$\arctan 1 = \frac{\pi}{4}$ so at $x=1$ $y' = \frac{1}{\sqrt{1+(\frac{\pi}{4})^2}} \cdot \frac{\pi}{4} \cdot \frac{1}{2}$.

(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?

arcsin z defined if $-1 \leq z \leq 1$

(4) (Final, 2016) An object is thrown straight up into the air at time $t = 0$ seconds. Its height in metres at time t seconds is given by $h(t) = s_0 + v_0 t - 5t^2$. In the first second the object rises by 5 metres. For how many seconds does the object rise before beginning to fall?

(5) A emergency breaking car can decelerate at $9\frac{\mathrm{m}}{\mathrm{s}^2}$. How fast can a car drive so that it can come to a stop within 50m?

if an object is at position $x(t)$, its velocity is $\frac{dx}{dt} = v(t)$ and its acceleration is $a(t) = \frac{dv}{dt} = \frac{d^2x}{dt^2}$. (speed $= |v|$)

2. Velocity and Acceleration

(3) A particle's position is given by $f(t) = t + 6e^{-t/3}$.

(a) Find the velocity at time t, and specifically at $t = 3$.

$v(t) = \frac{df}{dt} = 1 - 2e^{-t/3}$

$v(3) = 1 - \frac{2}{e}$

$\frac{2}{e^{t/3}}$

(b) When is the particle moving to the right? to the left?

$v(t) > 0$ if $1 - 2e^{-t/2} > 0 \Rightarrow 2e^{-t/3} < 1 \iff e^{t/3} > 2$

if $t > 3\log 2$, $v(t) < 0$ if $t < 3\log 2$ $\quad 2 < e^{t/3}$

(c) When is the particle accelerating? decelerating?

$a(t) = \frac{2}{3}e^{-t/3} > 0$ but need to see if $a(t)$, $v(t)$ have same or opposite signs

Related rates

- Similar to implicit diff: have a <u>relation</u> between two variables. Now diff wrt <u>another</u> variable (think of it as time)

<u>Example</u>: say point (x,y) is moving on the circle $x^2+y^2=1$ diff wrt t get:

$$2x\cdot\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

relation of 4 quantities $x, y, \frac{dx}{dt}, \frac{dy}{dt}$.

Given 3 can solve for the fourth

(only need 2 since also have original equation)

3. Related Rates

(6) A particle is moving along the curve $y^2 = x^3 + 2x$. When it passes the point $(1, \sqrt{3})$ we have $\frac{\mathrm{d}y}{\mathrm{d}t} = 1$. Find $\frac{\mathrm{d}x}{\mathrm{d}t}$.

$2y \frac{dy}{dt} = 3x^2 \frac{dx}{dt} + 2\frac{dx}{dt}$ at $(1, \sqrt{3})$:

$2\sqrt{3} \cdot 1 = 3 \frac{dx}{dt} + 2\frac{dx}{dt}$ so $\frac{dx}{dt} = \frac{2\sqrt{3}}{5}$

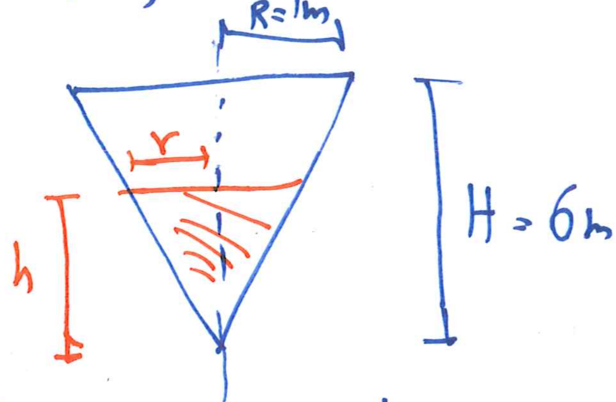
sometimes relation not given directly.

(7) (Final, 2015, variant) A conical tank of water is 6m tall and has radius 1m at the top.

(a) The drain is clogged, and is filling up with rainwater at the rate of $5\text{m}^3/\text{min}$. How fast is the water rising when its height is 5m?

recommendation read question, draw diagram

vertical cross-section:



give names: h = height of water

V = volume of water

r = radius of top of water

(b) The drain is unclogged and water begins to clear at the rate of $\frac{\pi}{4}\text{m}^3/\text{min}$ (but rain is still falling). At what height is the water falling at the rate of $1\text{m}/\text{min}$?