10. Logarithmic and Implicit differentiation (14/10/2021)

Goals.

(1) Differentiation involving logarithms
(2) Implicit differentiation

Last Time. Chain rule: $f(g(x))' = f'(g(x))\, g'(x)$

or: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$.

inverse fcn rule: if $f(g(y)) = y$,

$f'(f^{-1}(y))\,(f^{-1})'(y) = 1$

$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$

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if g is inverse to f: $\begin{cases} f^{-1}(f(x)) = x \\ f(f^{-1}(y)) = y \end{cases}$

Apply chain rule, get $\begin{cases} (f^{-1})'(f(x)) \cdot f'(x) = 1 \\ f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1 \end{cases}$

Math 100 – WORKSHEET 10
LOGARITHMIC AND IMPLICIT DIFFERENTIATION

1. Review of Logarithms

(1) $\log\left(e^{10}\right) = 10$ $\qquad$ $\log(2^{100}) = 100\log 2$

(2) A variant on *Moore's Law* states that computing power doubles every 18 months. Suppose computers today can do N_0 operations per second.

(a) Write a formula predicting the future:

- Computers t years from now will be able to do $N(t)$ operations per second where

$$N(t) = N_0 \cdot 2^{t/1.5} = N_0 \cdot e^{\frac{\log 2}{1.5} t}.$$

In t years see $t/1.5$ doublings

$2 = e^{\log 2}$

Exponential behaviour is

$C \cdot e^{kt}$ or $C \cdot q^t$

Date: 14/10/2021, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Differentiation

$$\boxed{(\log x)' = \frac{1}{x}}$$

(1) Differentiate

(a) $\frac{\mathrm{d}(\log(ax))}{\mathrm{d}x} =$

$= \frac{1}{ax} \cdot a = \frac{1}{x}$

or $\frac{d(\log(ax))}{dx} = \frac{d(\log x + \log a)}{dx} = \frac{1}{x} + 0$

$\frac{\mathrm{d}}{\mathrm{d}t} \log\left(t^2 + 3t\right) = \frac{2t+3}{t^2+3t}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 \log(1 + x^2) =$

product rule

$= 2x \cdot \log(1+x^2) + x^2 (\log(1+x^2))'$

$= 2x \log(1+x^2) + x^2 \cdot \frac{2x}{1+x^2}$

chain rule

$\frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\log(2+\sin r)} =$ chain rule

$-\frac{1}{(\log(2+\sin r))^2} \cdot \frac{1}{2+\sin r} \cdot \cos r$

Log converts product to sum

Goal: differentiate $y = \frac{x^2+1}{x+1} \cdot \frac{7+\cos x}{1+e^x}$,

For this, hit expression with log,

$$\log y = \log(x^2+1) + \log(7+\cos x) - \log(x+1) - \log(1+e^x)$$

Now diff. both sides wrt $\underline{x}$.

$$(\log y)' = \frac{1}{y} \cdot y' = \frac{2x}{x^2+1} - \frac{\sin x}{7+\cos x} - \frac{1}{x+1} - \frac{e^x}{1+e^x}.$$

Solve for y':

$$y' = \left(\frac{x^2+1}{x+1} \cdot \frac{7+\cos x}{1+e^x}\right)\left(\frac{2x}{x^2+1} - \frac{\sin x}{7+\cos x} - \frac{1}{x+1} - \frac{e^x}{1+e^x}\right)$$

want answer in terms of x if possible

(2) (Logarithmic differentiation) differentiate $y = \left(x^2+1\right) \cdot \sin x \cdot \frac{1}{\sqrt{x^3+3}} \cdot e^{\cos x}$.

$\frac{1}{\sqrt{x^3+3}} = (x^3+3)^{-\frac{1}{2}}$

$\log y = \log(x^2+1) + \log \sin x - \frac{1}{2}\log(x^3+3) + \cos x$

$\log(e^u) = u$

so $\frac{y'}{y} = \frac{2x}{1+x^2} + \frac{\cos x}{\sin x} - \frac{3x^2}{2(x^3+3)} - \sin x$

so $y' = (x^2+1)\sin x \frac{1}{\sqrt[2]{x^3+3}} e^{\cos x}\left(\frac{2x}{1+x^2} + \frac{1}{\tan x} - \frac{3x^2}{2(x^3+3)} - \sin x\right)$

Logarithmic differentiation:

$(\log f)' = \frac{f'}{f}$ so $f' = f \cdot (\log f)'$

good if f is a <u>pdt</u> or a <u>power</u>.

power law

<u>recall.</u> $(x^a)' = a \cdot x^{a-1}$

<u>Pf:</u> $(x^a)' = x^a \cdot (\log(x^a))' = x^a (a \log x)' = x^a \cdot a \cdot \frac{1}{x}$

exponential

<u>Also:</u> $(a^x)' = a^x \cdot \log a$

Using chain rule, can handle $f(x)^a$ or $a^{g(x)}$.
what about $f(x)^{g(x)}$? need to hit it with a log.

(3) Differentiate using $\boxed{f' = f \times (\log f)'}$

(a) x^x

$\log(x^x) = x \cdot \log x$ so $(\log(x^x))' = (x \log x)' = \log x + x \cdot \frac{1}{x}$

By log. diff. $(x^x)' = x^x(\log x + 1)$

or: let $y = x^x$ then $\log y = x \log x$ so $y'/y = \log x + 1$ so $y' = y(\log x + 1)$

$\boxed{= x^x(\log x + 1)}$

(b) $(\log x)^{\cos x}$

$$((\log x)^{\cos x})' = (\log x)^{\cos x} \cdot (\log((\log x)^{\cos x}))' = (\log x)^{\cos x}(\cos x \cdot \log \log x)'$$

$$= (\log x)^{\cos x}\left(-\sin x \cdot \log \log x + \cos x \cdot \frac{1}{\log x} \cdot \frac{1}{x}\right).$$

(c) (Final, 2014) Let $y = x^{\log x}$. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of x only.

$\log y = (\log x) \cdot (\log x) = (\log x)^2$ so $\frac{y'}{y} = (\log y)' = 2 \log x \cdot \frac{1}{x}$

so $y' = 2 \log x \cdot \frac{1}{x} \cdot x^{\log x} = 2 \log x \cdot x^{\log x - 1}.$

Implicit differentiation

Idea: if I have a relation between x, y I can calculate $\frac{dy}{dx}$ ~~along the curve~~ along the curve without solving for y.

Ex: say $y + e^{xy} = x$ can find y' in terms of both x, y.

diff both sides w.r.t. x: get

$$y' + e^{xy}(1 \cdot y + x y') = 1$$

$$\Rightarrow y'(1 + x e^{xy}) + e^{xy} y = 1$$

so $$y' = \frac{1 - e^{xy} y}{1 + x e^{xy}}$$

e.g. in the point $x = 0$, $y = -1$ have

$$y' = \frac{1 - e^0 \cdot (-1)}{1 + 0} = 2$$