9. The Chain Rule; Inverse functions (12/10/2021)

Goals.

(1) Composition of functions
(2) The chain rule
(3) The inverse function rule

Last Time.

$$\frac{d}{dx}(9^x) = (\log 9)\, 9^x \;;\quad \frac{d(\sin\theta)}{d\theta} = \cos\theta$$

$$\frac{d(\tan\theta)}{d\theta} = 1+\tan^2\theta = \frac{1}{\cos^2\theta} \;;\quad \frac{d(\cos\theta)}{d\theta} = -\sin\theta$$

<u>Example</u>: $\left(\frac{\sqrt{x}}{9^x}\right)'$

<u>observe</u>: this is a quotient, numerator is a power law denominator is an exponential.

but also

$$\frac{\sqrt{x}}{9^x} = \sqrt{x}\cdot 9^{-x} = \sqrt{x}\left(\frac{1}{9}\right)^x$$

f is the <u>composition</u> of g and h if

$$f(x) = g(h(x))$$

<u>Suggestion:</u> write $g(x)$ not $g(x)$

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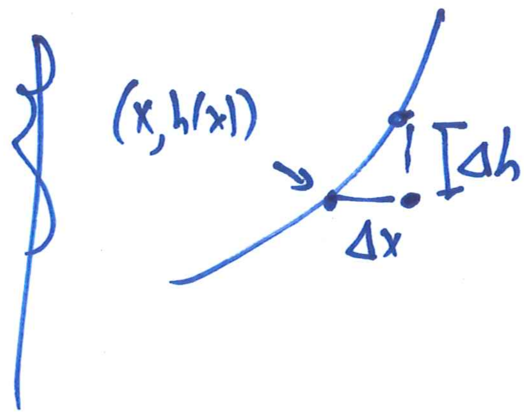
move from x to $x + \Delta x$

then $h(x)$ changes to $h(x) + \Delta h$

with $\Delta h \approx h'(x)\Delta x$



$$\Delta f \approx g'(h(x))\,\Delta h$$

$$\approx g'(h(x)) \cdot h'(x) \cdot \Delta x$$

$$x \xrightarrow{h} h(x) \xrightarrow{g} g(h(x))$$ ← chain

<u>chain rule:</u>

$$\frac{df}{dx} = g'(h(x)) \cdot h'(x)$$

$$= \frac{dg}{dh} \cdot \frac{dh}{dx}$$

Math 100 – WORKSHEET 9
THE CHAIN RULE; INVERSE FUNCTIONS

1. THE CHAIN RULE

(1) Write the function as a composition and then differentiate.

(a) e^{3x}

① let $g(u) = e^u$, $h(x) = 3x$, so that $e^{3x} = g(h(x))$

then $(e^{3x})' = (e^u)' \cdot (3x)' = e^u \cdot 3 = 3e^{3x}$

alternative

② $e^{3x} = e^u$ where $u = 3x$ so $\frac{d(e^{3x})}{dx} = \frac{d(e^u)}{du} \cdot \frac{du}{dx} = e^u \cdot 3 = 3e^{3x}$.

Either way go back to fcn of x.

(b) $\sqrt{2x+1}$

$\sqrt{2x+1} = \sqrt{u}$ with $u = 2x+1$

so $\frac{d(\sqrt{2x+1})}{dx} = \frac{d(\sqrt{u})}{du} \cdot \frac{du}{dx} = \frac{1}{2\sqrt{u}} \cdot 2 = \frac{1}{\sqrt{2x+1}}$

Date: 12/10/2021, Worksheet by Lior Silberman.

(c) (Final, 2015) $\sin(x^2)$

$\sin(x^2) = g(h(x)) \qquad g(u) = \sin u$

$\qquad\qquad\qquad\qquad\quad h(x) = x^2$

so $\quad g(h(x))' = g'(h(x)) \cdot h'(x) = \cos(x^2) \cdot 2x$

(d) $(7x + \cos x)^n$.

$$\frac{d((7x+\cos x)^n)}{dx} = n\,(7x+\cos x)^{n-1} \cdot (7 - \sin x)$$

in detail, let $u = 7x + \cos x$.

then $$\frac{d((7x+\cos x)^n)}{dx} = \frac{d(u^n)}{du} \cdot \frac{du}{dx} = nu^{n-1} \cdot (7 - \cancel{\cos}\,\sin x)$$

$$= n\,(7x+\cos x)^{n-1} \cdot (7 - \sin x)$$

(2) (Final, 2012) Let $f(x) = g(2\sin x)$ where $g'(\sqrt{2}) = \sqrt{2}$. Find $f'\left(\frac{\pi}{4}\right)$.

If $f(x) = g(2\sin x)$ then $f'(x) = g'(2\sin x)\cdot(2\cos x)$

$\left[\begin{array}{l}\text{let } u = 2\sin x \\ \text{then } f'(x) = g'(u)\cdot\frac{du}{dx}\end{array}\right\}$ or: $f(x) = g(h(x))$, $h(x) = 2\sin x$ $\Big]$

Now $f'\left(\frac{\pi}{4}\right) = g'\left(2\cdot\sin\frac{\pi}{4}\right)\cdot\left(2\cos\frac{\pi}{4}\right) = g'\left(2\cdot\frac{1}{\sqrt{2}}\right)\cdot 2\cdot\frac{1}{\sqrt{2}}$

g takes h-values not x-values

$\sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\frac{2}{\sqrt{2}} = \frac{(\sqrt{2})^2}{\sqrt{2}} = \sqrt{2}$

$= g'(\sqrt{2})\cdot\sqrt{2} = \sqrt{2}\cdot\sqrt{2} = 2$

(3) Differentiate

(a) $7x + \cos(x^n)$

$$(7x + \cos(x^n))' = 7 - \sin(x^n) \cdot nx^{n-1}$$

$$(7x + \cos(x^n))' = 7(x)' + (\cos(x^n))' = 7 + (-\sin(x^n) \cdot nx^{n-1})$$

linearity

chain rule

$\cos(x^n) = \cos(u),\ u = x^n$

(b) $e^{\sqrt{\cos x}}$

$$e^{\sqrt{\cos x}} = e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} (-\sin x)$$

or

$$\frac{d}{dx} e^{\sqrt{\cos x}} = \frac{d(e^{\sqrt{\cos x}})}{d(\sqrt{\cos x})} \cdot \frac{d(\sqrt{\cos x})}{dx} = e^{\sqrt{\cos x}} \cdot \frac{d(\sqrt{\cos x})}{d(\cos x)} \cdot \frac{d\cos x}{dx} =$$

$$= e^{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot -\sin x$$

(c) (Final 2012) $e^{(\sin x)^2}$

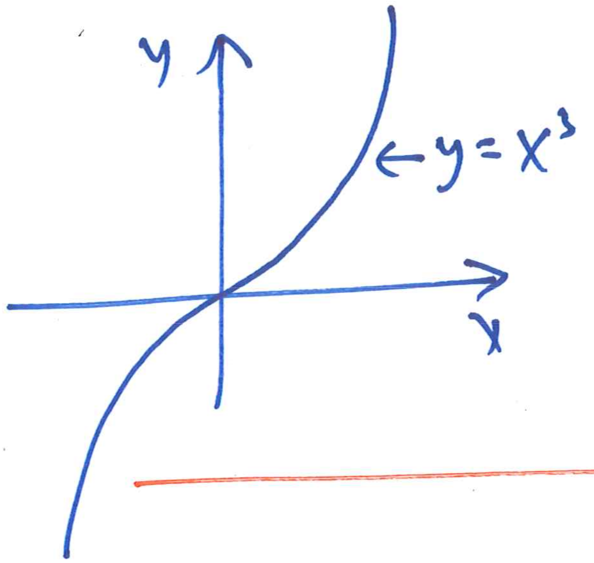
$$(e^{(\sin x)^2})' = e^{(\sin x)^2} \cdot 2\sin x \cdot \cos x$$

or: $e^{(\sin x)^2} = e^u,\ u = (\sin x)^2 \qquad u = v^2,\ v = \sin x$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx} = e^u \cdot 2v \cdot \cos x = e^{(\sin x)^2} \cdot 2\sin x \cos x$$

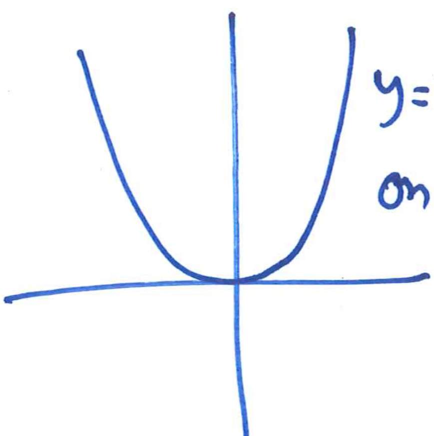
Inverse function:

If $y = f(x)$ the inverse $x = f^{-1}(y)$ is the x value s.t. $y = f(x)$



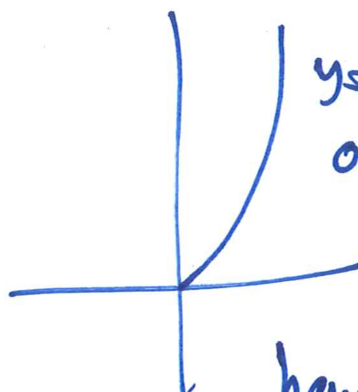
$y = x^3$ has an inverse (it's increasing) which is

$x = y^{1/3}$



$y = x^2$ on $(-\infty, \infty)$

no inverse: two x-values for each y value



$y = x^2$ on $[0, \infty)$

have an inverse

$x = y^{\frac{1}{2}} = \sqrt{y}$

defined for $y \geq 0$

(range of $y = x^2$)

Example. $f(x) = 2 + x + \sin x$

$f^{-1}(2+\pi) = \pi$ since $f(\pi) = 2 + \pi + \sin \pi = 2 + \pi$

(4) Suppose f, g are differentiable functions with $f(g(x)) = x^3$. Suppose that $f'(g(4)) = 5$. Find $g'(4)$.

Since $f(g(x)) = x^3$, we have $f'(g(x)) \cdot g'(x) = 3x^2$,

thus $f'(g(4)) \cdot g'(4) = 3 \cdot 4^2$

so $5 \cdot g'(4) = 48$, so $\boxed{g'(4) = \frac{48}{5}}$

2. Inverse functions

(5) Find the function inverse to $y = x^7 + 3$.

If $y = x^7 + 3$ then $x^7 = y - 3$ so $x = (y-3)^{1/7}$

(for odd powers, $y^{\frac{1}{7}}, y^{\frac{1}{3}}, \ldots$, allow negative y-values)

(6) Does $y = x^2$ have an inverse?

(7) Consider the function $y = \sqrt{x-1}$ on $x \geq 1$.

(a) Find the inverse function, in the form $x = g(y)$.

~~then~~ here $x = y^2 + 1$

so $\frac{dx}{dy} = 2y$, $\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}$

so $\frac{dx}{dy} \cdot \frac{dy}{dx} = \frac{2y}{2\sqrt{x-1}} = 1$

(b) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$ and calculate their product.

Inverse function rule: If $f(g(x)) = x$

then $f'(g(x)) \cdot g'(x) = 1$

or $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ along $y = f(x)$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

(8) Let $f(x) = \log x$. Apply the chain rule to the formula $f(e^y) = y$ to get a formula for $f'(e^y)$, and use that to determine the derivative of the logarithm.

If $y = \log x$, $x = e^y$ so $\frac{dx}{dy} = e^y$

so $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$ i.e. $\frac{d(\log x)}{x} = \frac{1}{x}$.

(9) Let $f(x) = x^3 + 5x$. Find $f^{-1}(6)$ and $\left(f^{-1}\right)'(6)$.

$f^{-1}(6) = 1$ since $f(1) = 1^3 + 5 \cdot 1 = 6$

$(f^{-1})'(6) = \frac{1}{f'(1)}$

inverse fcn rule

$$\boxed{(f^{-1})'(y) = \frac{1}{f'(x)} \text{ if } y = f(x)}$$