2. Limit laws (14/9/2021)

Goals.

(1) Limits, again
(2) Piecewise-defined functions
(3) Existence and nonexistence of limits: blowup
(4) Limit laws
(5) The squeeze/sandwich theorem

Last Time.

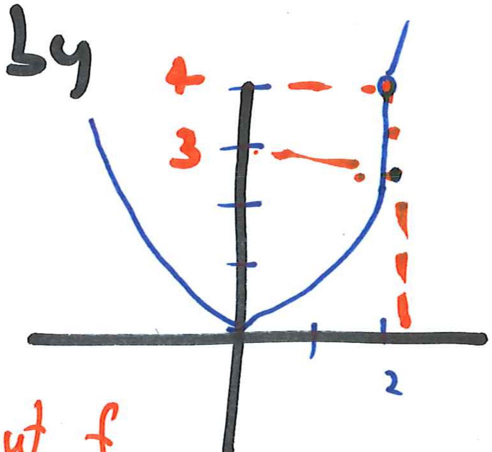
limiting processes
$\rightsquigarrow$ limits.

No lecture Thursday

Watch recording

Example: Let f be given by

$$f(x) = \begin{cases} x^2 & x \neq 2 \\ 3 & x = 2 \end{cases}$$



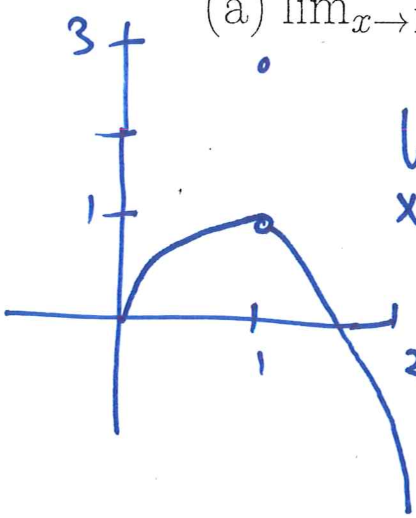
$\lim_{x \to 2} f(x) = 4$ ← limit is about f near 2, not at 2.

Math 100 – WORKSHEET 2
LIMIT LAWS

1. Existence of limits and blowup

(1) Either evaluate the limit or explain why it does not exist. Sketching a graph might be helpful.

(a) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 3 & x = 1 \\ 2 - x^2 & x > 1 \end{cases}$.



$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x} = \sqrt{1} = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2-x^2) = 2-1^2 = 1$

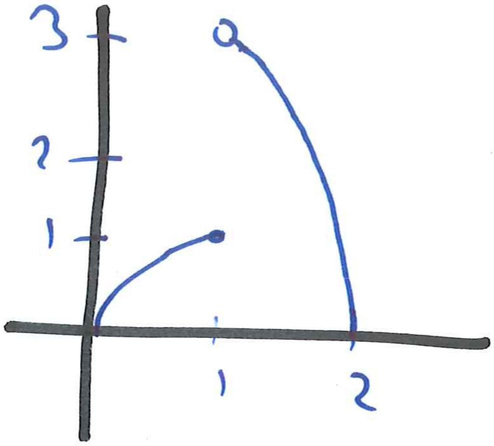
so $\lim_{x\to 1} f(x)$ exists, equals 1.

(b) $\lim_{x\to 1} f(x)$ where $f(x) = \begin{cases} \sqrt{x} & 0 \le x < 1 \\ 1 & x = 1 \\ 4 - x^2 & x > 1 \end{cases}$.

$\lim_{x\to 1^-} f(x) = 1$ (same as above)

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (4-x^2) = 4-1^2 = 3$

$1 \neq 3$ so $\lim_{x\to 1} f(x)$ does not exist (DNE)

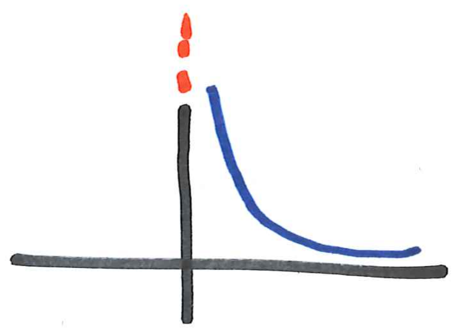


Reasons for no limit:

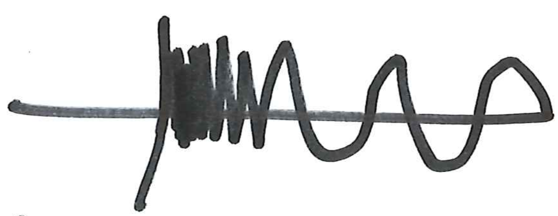
(1) jump: one-sided limits disagree

(2) blowup: values become ∞-ly large.

E.g. $\lim\limits_{x \to 0^+} \frac{1}{x}$



(3) oscillation: E.g.



$f(x) = \sin\left(\frac{1}{x}\right)$ if $x > 0$

(2) Let $f(x) = \frac{x-3}{x^2+x-12}$.

(a) (Final 2014) What is $\lim_{x\to 3} f(x)$?

$$\left[\frac{3-3}{3^2+3-12} \text{ "="} \frac{0}{0}\right] \quad \text{if } x \neq 3 \quad \frac{x-3}{x^2+x-12} = \frac{\cancel{x-3}}{(\cancel{x-3})(x+4)} = \frac{1}{x+4} \xrightarrow{x \to 3} \frac{1}{7}$$

(b) What about $\lim_{x\to 2} f(x)$?

$$\lim_{x\to 2} \frac{x-3}{x^2+x-12} = \frac{(\lim_{x\to 2} x) - 3}{(\lim_{x\to 2} x)^2 + (\lim_{x\to 2} x) - 12} = \frac{2-3}{2^2+2-12} = \frac{-1}{-6} = \frac{1}{6}.$$

Error:

$$\lim_{x\to 3} \frac{x-3}{x^2+x-12} = \frac{1}{x+4} = \frac{1}{7}$$

$$\frac{x-3}{x^2+x-12} = \frac{2-3}{2^2+2-12} = \frac{1}{6},$$

(instead: $\lim_{x\to 2} \frac{x-3}{x^2+x-12} = \frac{1}{6}$ or $\frac{x-3}{x^2+x-12} \xrightarrow[x\to 2]{} \frac{1}{6}$

Fact: $\sqrt{a} - \sqrt{b} = \sqrt{a} - \sqrt{b} \cdot \frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$

(5) Evaluate:

(a) $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x}$.

$$\frac{\sqrt{4+x}-2}{x} = \frac{(4+x)-4}{\sqrt{4+x}+2} \cdot \frac{1}{x} = \frac{x}{x(\sqrt{4+x}+2)} \overset{x\neq 0}{=}$$

$$= \frac{1}{\sqrt{4+x}+2} \xrightarrow[x\to 0]{} \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

or: $\lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x\to 0} \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2} = \ldots = \lim_{x\to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1+x^2}}{x}$.

$$\frac{\sqrt{1+x}-\sqrt{1+x^2}}{x} = \frac{(1+x)-(1+x^2)}{x(\sqrt{1+x}+\sqrt{1+x^2})} = \frac{x-x^2}{x(\sqrt{\ }+\sqrt{\ })} =$$

$$\overset{x\neq 0}{=} \frac{1-x}{\sqrt{1+x}+\sqrt{1+x^2}} \xrightarrow[x\to 0]{} \frac{1-0}{\sqrt{1}+\sqrt{1}} = \frac{1}{2}.$$

challenge: $\lim\limits_{x\to 3} f(x) = 2$, $\lim\limits_{x\to 3} g(x) = 7$,

$\lim\limits_{x\to 3} h(x) = -2$ What is

$$\lim_{x\to 3} \frac{f(x)+g(x)}{f(x)\cdot g(x)+h(x)} ?$$

$$\lim_{x\to 3} \frac{f(x)+g(x)}{f(x)g(x)+h(x)} = \frac{(\lim\limits_{x\to 3} f(x)) + (\lim\limits_{x\to 3} g(x))}{(\quad)(\quad)+(\lim\limits_{x\to 3} h(x))} =$$

$$= \frac{2+7}{2\cdot 7-2} = \frac{9}{12} = \frac{3}{4}.$$