Lior Silberman's Math 223: Problem Set 9 (due 22/3/2021)

Hint for 1,2,3: if you aren't sure try what happens with small matrices ($2\times 2, 3\times 3, 4\times 4, 5\times 5$) before tackling the general case.

Three determinants

1. Fix numbers a,b and let H_n be the matrix with entries t_{ij} so that for all i, $t_{ii}=a$, $t_{i,(i-1)}=t_{i,(i+1)}=b$ and $t_{ij}=0$ otherwise. Let $h_n=\det H_n$.
   (a) For $n\geq 1$ show that $h_{n+2}=ah_{n+1}-b^2h_n$.
   (b) Using the method of problem 5 below solve the recursion in the case $a=5$, $b=2$ and find a closed-form expression for h_n.

2. Let $H_n(d_1,\cdots,d_n)$ be the matrix $J_n+\mathrm{diag}(d_1,\ldots,d_n)$ where J_n is the all-ones matrix and let $h_n(d_1,\cdots,d_n)=\det[H_n(d_1,\cdots,d_n)]$.
   (a) Show that $h_n(0,d_2,\ldots,d_n)=\prod_{j=2}^n d_j$. (Hint: subtract the second row from the first)
   (b) Suppose that $n\geq 2$. Show that $h_n(d_1,d_2,\ldots,d_n)=d_1h_{n-1}(d_2,\ldots,d_n)+d_2h_{n-1}(0,d_3,\ldots,d_n)$.
   (c) Suppose that all the $d_i\neq 0$ and that $n\geq 2$. Show that $\frac{h_n(d_1,\ldots,d_n)}{\prod_{j=1}^n d_j}=\frac{h_{n-1}(d_2,\ldots,d_n)}{\prod_{j=2}^n d_j}+\frac{1}{d_1}$.
   (d) Show that $\frac{h_2(d_1,d_2)}{d_1d_2}=\frac{1}{d_1}+\frac{1}{d_2}+1$, and thus that $\frac{h_n(d_1,\ldots,d_n)}{\prod_{j=1}^n d_j}=\sum_{j=1}^n\frac{1}{d_j}+1$.
   CONCLUSION $h_n(d_1,\ldots,d_n)=\left(\sum_{j=1}^n\frac{1}{d_j}+1\right)\left(\prod_{j=1}^n d_j\right)$.

3. (The "Vandermonde determinant") Let x_i be variables and let $V_n(x_1,\ldots,x_n)$ be the $n\times n$ matrix with entries $v_{ij}=x_i^{j-1}$. We show that $\det V_n=\prod_{i=2}^n\prod_{j=1}^{i-1}\left(x_i-x_j\right)$.
   (a) Show that $\det V_n$ is a polynomial in $x_1,\ldots,x_n$ of total degree $0+1+2+3+\cdots+(n-1)=\frac{n(n-1)}{2}$.
   (b) Show that $\det V_n$ vanishes whenever $x_i=x_j$ (which leads you to suspect that x_i-x_j divides the polynomial).
   RMK Note that $\prod_{i=2}^n\prod_{j=1}^{i-1}\left(x_i-x_j\right)$ is a polynomial of total degree $\frac{n(n-1)}{2}$. It follows from (a) and the theory of polynomial rings over integral domains that $\prod_{i=2}^n\prod_{j=1}^{i-1}\left(x_i-x_j\right)$ actually does divide the determinant, and comparing degrees of the two it follows that the quotient has degree zero, that is that for some constant $c_n\in\mathbb{Z}$, $\det V_n=c_n\prod_{i=2}^n\prod_{j=1}^{i}\left(x_i-x_j\right)$.
   SUPP Examining the coefficient of $x_1^0x_2^1x_3^2\cdots x_n^{n-1}$ show that $c_n=1$.
   (d) Let $V_{n+1}(x_1,\ldots,x_{n+1})$ be the matrix described above, and let W_{n+1} be the matrix obtained by
      (i) Subtracting the first row from each row; and then
      (ii) For j descending from $n+1$ to 2, subtracting from the jth column a multiple of the $(j-1)$st so as to make the top entry in the column zero.
   Let $\left(w_{ij}\right)_{i,j=1}^{n+1}$ be the entries of W_{n+1}. Show that $w_{11}=1$ that $w_{1j}=w_{i1}=0$ if $i,j\neq 1$ and that $w_{ij}=(x_i-x_1)v_{i,j-1}$ if $i,j\geq 2$.
   (e) Show that $\det V_{n+1}=\left[\prod_{i=2}^{n+1}(x_i-x_1)\right]\cdot[\det V_n(x_2,\ldots,x_{n+1})]$.
   (f) Check that $\det V_1=1$ and prove the main claim by induction.

SUPP (Polynomial interpolation) Let $\{(x_i,y_i)\}_{i=1}^k\subset\mathbb{R}^2$ be points in the plane with distinct x_i. Show that there exists a unique polynomial $p\in\mathbb{R}[x]^{<k}$ such that $p(x_i)=y_i$.

Linear recurrences

4. Let $T \in \text{End}(V)$ and let $\underline{v} \in V$ satisfy $T\underline{v} = \lambda \underline{v}$.
   (a) Show that $T^n \underline{v} = \lambda^n \underline{v}$ for all $n \geq 0$.
   (b) Suppose that T is invertible and $\underline{v} \neq 0$. Show that $\lambda \neq 0$ and that $T^{-n}\underline{v} = \lambda^{-n}\underline{v}$.
   (c) Let $p \in \mathbb{R}[x]$ be a polynomial of degree n. Show that $p(T)\underline{v} = p(\lambda)\underline{v}$, where $p(T)$ is the linear map defined in the supplement to PS6.

5. A sequence $\underline{F} \in \mathbb{C}^{\mathbb{N}}$ satisfies a *recursion relation of degree k if we have coefficients $c_0, \ldots, c_{k-1}$ such that* $F_{n+k} = \sum_{i=0}^{k-1} c_i F_{n+i}$ for all $n \geq 0$. In that case let $p(x) = x^k - \sum_{i=0}^{k-1} c_i x^i$ be the *characteristic polynomial* of the recursion relation.
   (a) Explain why we generally assume $c_0 \neq 0$.
   (b) Show that $\underline{F}$ satisfies the recursion relation iff $p(L)\underline{F} = \underline{0}$, were $L \in \text{End}(\mathbb{R}^\infty)$ is the left shift.
   (c) Show that $\text{Ker}(p(L))$ is k-dimensional, and that any $\underline{F} \in \text{Ker}(p(L))$ is determined by $(F_0, F_1, \ldots, F_{k-1})$.
   (d) Suppose that r is a root of $p(x)$. Show that the sequence $(r^n)_{n\geq 0} \in \text{Ker}(p(L))$ and that it is non-zero.

   FACT A set of (non-zero) eigenvectors corresponding to distinct eigenvalues is linearly independent.
   ASSUME for the rest of the problem that $p(x)$ has k distinct roots $\{r_i\}_{i=1}^{k}$.
   (e) Find a basis for $\text{Ker}(p(L))$.
   (f) Let $(F_0, F_1, \ldots, F_{k-1})$ be any numbers. Show that the system of k equations $\sum_{i=0}^{k-1} A_i r_i^j = F_j$ $(1 \leq j \leq k)$ in the unknowns A_i has a unique solution. (Hint: problem 3)

6. Practice with complex numbers
   (a) Let $w = a + bi$ be a non-zero complex number. Show that there are two complex solutions to the equation $z^2 = w$. (Hint: write $z = x + yi$ and get a system of two equations in the unknowns x, y).
   (b) Let $a, b, c \in \mathbb{C}$ with $a \neq 0$. Show that the polynomial $az^2 + bz + c \in \mathbb{C}[z]$ factors as a product of linear polynomials. (Hint: use the quadratic formula)

Challenge: Practice with Incidence geometry

An *incidence structure* is a triple pair $(P, L, \in)$ where P is a set (its elements are called *points*), L is a set (its elements are called "lines"), and $\in$ is a relation between the sets P, L. We interpret the situation $p \in \ell$ as "the point p lies on the line ℓ" (is incident to it) and $p \notin \ell$ to be the reverse sitaution. We always assume that P, L are finite. Our goal is to prove

THEOREM (De Bruin–Erdős). *Suppose that for any two distinct points p, p' there is a unique line ℓ such that $p \in \ell$ and $p' \in \ell$, and that not all points are on the same line. Then there are at least as many lines as points.*

*7. Let $(P, L, \in)$ be an incidence structure which satisfies the axiom: "any two distinct points are incident to a unique line".
   (a) Suppose that for some point p there is only one line containing p. Show that this line contains all points.

   DEF Let $T\colon \mathbb{R}^P \to \mathbb{R}^L$, $S\colon \mathbb{R}^L \to \mathbb{R}^P$ be the maps $(Tf)(\ell) = \sum_{p\in\ell} f(p)$ (sum over points on ℓ) and $(Sg)(p) = \sum_{p\in\ell} g(\ell)$ (sum over lines containing p).
   (b) Show that T, S are linear.
   (c) Suppose that $P = \{p_i\}_{i=1}^n$ is finite. Show that the matrix of ST in the "standard basis" of $\mathbb{R}^P$ (the ith basis vector is the function which is 1 at p_i, zero elsewhere) is $J_n + \text{diag}(d_1 - 1, \ldots, d_n - 1)$ where J_n is the all-ones matrix and d_i is the number of lines through p_i.
   (d) Suppose that not all points are on the same line. Show that $\det(ST) > 0$.
   (e) Prove the Theorem.

8. Suppose that we add the axiom "every two distinct lines intersect at exactly one point".
   (a) Show that in this case exchanging the role of points and lines (and the adjusting the relation appropriately) gives a new incidence structure (the "dual one") which also satisfies both the axiom of problem 7 and the axiom we just introduced.
   (b) Conclude that with the extra axiom there only three possibilities: (1) there is exactly one line and it contains all the points; (2) there is exactly one point and it lies on all lines; (3) there are as many lines as points

Supplementary problem: Quadratic extensions in general

A (Constructing quadratic fields) Let F be a field, $d \in F$ such that $x^2 = d$ has no solutions in F.
   (a) Show that the set of matrices $E = \left\{ \begin{pmatrix} a & b \\ db & a \end{pmatrix} \mid a,b \in F \right\}$ is a two-dimensional F-subspace of $M_2(F)$ with basis $1, \varepsilon$, where $\varepsilon = \begin{pmatrix} & 1 \\ d & \end{pmatrix}$ satisfies $\varepsilon^2 = d$.
   (b) Show that E is also closed under matrix multiplication and transpose.
   (c) Show that the map $\sigma \colon E \to E$ given by $\sigma(x) = x^t$ satisfies $\sigma(x+y) = \sigma(x) + \sigma(y)$, $\sigma(xy) = \sigma(x)\sigma(y)$, $\sigma(a+b\varepsilon) = a - b\varepsilon$ for all $x,y \in E$, $a,b \in F$.
   (d) Show that the norm $Nz = z\sigma(z)$ satisfies $Nz \in F$ for all $z \in E$, $Nz \neq 0$ if $z \neq 0$, $N(zw) = NzNw$.
   (e) Conclud that E is a field.

B. (Uniqueness) Let E' be a field containing F which is two-dimensional over F.
   (a) Suppose E' is spanned over F by elements $1, \varepsilon$ with $\varepsilon^2 = d$. Let $z = a + b\sqrt{d} \in E'$ be any element and let $M_z \colon E' \to E'$ be the map of multiplication by z. Show that M_z is F-linear and that its matrix in the basis $\{1, \varepsilon\}$ is $\begin{pmatrix} a & b \\ db & a \end{pmatrix}$.
   (b) Show that E always has a basis of the form $\{1, \delta\}$ with $\delta \notin F$. Show that if $\operatorname{char} F \neq 2$ there is $\varepsilon = a + b\delta$ such that $\varepsilon^2 \in F$.
   (c) Show that $E = F(\sqrt{d})$ and $E' = F(\sqrt{d'})$ are isomorphic as fields iff $\frac{d}{d'}$ is a square in F.