

Lior Silberman's Math 223: Problem Set 7 (due 8/3/2021)**Practice problems (recommended, but do not submit)**

Section 3.2, Problems 1-8.

Section 3.3, Problems 1-8, 10.

Section 3.4, Problems 1-8

Gaussian elimination

1. Solve the following systems of linear equations using Gaussian elimination. Follow the following procedure: (1) Write the augmented matrix (2) note every step of your calculation (e.g. "add 2x the third row to the first") (3) Once you reach reduced row echelon form read off the solution. (4) Circle every pivot in the original matrix from step 1. If you swapped rows this need not be at the location you used it.

$$(a) \begin{cases} 2x + 3y = 5 \\ 3x + 5y = 7 \end{cases} \quad (b) \begin{cases} 2x + 3y + 5z = 2 \\ x + 2y - z = 1 \end{cases} \quad (c) \begin{cases} 2x + 3y + 4z = 7 \\ x - 2y + z = 5 \\ 7y + 2z = 3 \end{cases}$$

2. For each of the following matrices use Gaussian elimination to decide if it is invertible, and to construct an inverse if the matrix is invertible.

$$(a) A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \quad (b) B = \begin{pmatrix} 8 & 3 \\ 16 & 6 \end{pmatrix} \quad (c) C = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 7 & 3 \\ 1 & 7 & 2 \end{pmatrix}.$$

PRAC Let $A \in M_{n,m}(\mathbb{R})$ be a matrix. The *column-rank* of A is the dimension of the subspace of \mathbb{R}^n spanned by the columns of A . The *row-rank* is defined similarly.

- (a) Show in detail that the column-rank of A is equal to *its rank* $r(A) = \dim \text{Im } L_A$ where $L_A: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is the map of multiplication by A (we mentioned this in class).
- (b) Show that elementary row operations do not change the column-rank of a matrix. (Hint: show that composition with an invertible map does not change the rank of a linear transformation).
- (c) Show that elementary row operations do not change the row-rank of a matrix. (Hint: start by showing that they do not increase it, and then then use the reversibility of row operations).
- (d) Show that the row-rank of a matrix in reduced row echelon form is equal to the number of non-zero rows.
- (e) Conclude: the row-rank and column-rank are always both equal to the rank.

Raising matrices to powers

3. (powers of diagonal and diagonalizable matrices)

PRAC Let $A = \text{diag}(a_1, \dots, a_n), B = \text{diag}(b_1, \dots, b_n) \in M_n(\mathbb{R})$. Show that $A + B = \text{diag}(a_1 + b_1, \dots, a_n + b_n)$ and that $AB = \text{diag}(a_1 b_1, \dots, a_n b_n)$.

- (a) Let $A = \text{diag}(a_1, \dots, a_n) \in M_n(\mathbb{R})$ be diagonal, and let $P(k)$ be the statement " $A^k = \text{diag}(a_1^k, \dots, a_n^k)$ ". Show that $P(0)$ holds and that if $P(k)$ is true then so is $P(k+1)$. From the principle of mathematical induction it follows that $P(k)$ is true for all k .

(b) Use the identity $\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 3 & -2 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 26 & -18 \\ 36 & -25 \end{pmatrix}$ to calculate $\begin{pmatrix} 26 & -18 \\ 36 & -25 \end{pmatrix}^{1000}$.

4. Let F_n be the sequence defined by $F_0 = a, F_1 = b$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

(a) Show that $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} F_{n+1} \\ F_{n+2} \end{pmatrix}$ holds for all n .

(b) Show that $\begin{pmatrix} F_n \\ F_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n \begin{pmatrix} a \\ b \end{pmatrix}$ (hint: use the same scheme as in 3(a)).

RMK $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is similar to a diagonal matrix; the idea of 3(b) will give a simple formula for F_n .

Determinant calculations

5. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$, $C = \begin{pmatrix} 3 & 5 & 7 \\ 4 & 5 & 6 \\ 7 & 1 & 9 \end{pmatrix}$, $D = BC$.

(a) Evaluate the determinant of the above matrices using the definition. Check that $\det(BC) = \det B \cdot \det C$.

(b) Evaluate the same determinants using Gaussian elimination.

6. A *permutation matrix* is an $n \times n$ matrix P having all entries zero except that in every row and column there is exactly one 1. Examples include I_n , $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. Show that $\det P \in \{\pm 1\}$ for every permutation matrix.

Supplement: The Fibonacci sequence, again

Recall our notation $\mathbb{R}^\infty = \mathbb{R}^{\mathbb{N}}$ for the space of sequences, and let $L, R \in \text{End}(\mathbb{R}^\infty)$ be the “shift left” and “shift right” maps:

$$\begin{aligned} L(a_0, a_1, a_2, \dots) &= (a_1, a_2, \dots) \\ R(a_0, a_1, a_2, \dots) &= (0, a_0, a_1, a_2, \dots) \end{aligned}$$

that is,

$$\begin{aligned} (L\underline{a})_n &= a_{n+1} \\ (R\underline{a})_n &= \begin{cases} 0 & n = 0 \\ a_{n-1} & n \geq 1 \end{cases} \end{aligned}$$

A. (Basics)

(a) Find the kernel and image of L , concluding that it is surjective but not injective.

(b) Find the kernel and image of R , concluding that it is injective but not surjective.

(c) Show that $LR = \text{Id}$ but that $RL \neq LR$.

B. Let F_n be the sequence defined by $F_0 = a$, $F_1 = b$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$.

(a) Show that $(L^2 - L - 1)\underline{F} = \underline{0}$.

(b) Show that the map $\Phi: \text{Ker}(L^2 - L - 1) \rightarrow \mathbb{R}^2$ given by $\Phi(\underline{F}) = \begin{pmatrix} F_0 \\ F_1 \end{pmatrix}$ is an isomorphism of vector spaces.

Challenge problem

C. Show that the set $\{R^k L^l \mid k, l \geq 1\} \subset \text{End}(\mathbb{R}^\infty)$ is linearly independent.