# Lior Silberman's Math 223: Problem Set 5 (due 22/2/2021)

## Calculations with matrices

1. Let $A=\begin{pmatrix}-2 & 3\\ 5 & -7\end{pmatrix}$, $B=\begin{pmatrix}4 & 1 & 0\\ 0 & -2 & 9\end{pmatrix}$, $C=\begin{pmatrix}0 & 0\\ 1 & 1\\ 2 & 2\end{pmatrix}$, $D=\begin{pmatrix}7 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 5\end{pmatrix}$. Calculate all possible products among pairs of $A,B,C,D$ (don't forget that $A^2=AA$ is also such a product and that $XY,YX$ are different products if both make sense).

PRAC The $n\times n$ *identity matrix* is the matrix $I_n\in M_n(\mathbb{R})$ with entries: $(I_n)_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$. Show that $I_n\underline{v}=\underline{v}$ for all $\underline{v}\in\mathbb{R}^n$.

2. Let $A\in M_{m,n}(\mathbb{R})$. Show that $AI_n=I_mA=A$. (Hint)

PRAC
- (a) Let $A\in M_{n,m}(\mathbb{R})$, $B\in M_{m,p}(\mathbb{R})$. Show that the $j$th column of $AB$ is given by the product $A\underline{v}$ where $\underline{v}$ is the $j$th column of $B$.
- (b) Let $A\in M_{n,m}(\mathbb{R})$, $B\in M_{m,p}(\mathbb{R})$. Show that the $j$th column of $AB$ is a linear combination of all the columns of $A$ with the coefficients being the $j$th column of $B$.

3. Let $A,B\in M_n(\mathbb{R})$ be square matrices. We say $A,B$ *commute* if $AB=BA$. WE say $A$ is *scalar* if $A=zI_n$ for some $z\in\mathbb{R}$. The *centre* of $M_n(\mathbb{R})$ is the set $Z=\{A\in M_n(\mathbb{R})\mid\forall B\in M_n(\mathbb{R}):AB=BA\}$ of matrices that commute with all other matrices.
PRAC Check that the action of $zI_n$ on vectors is by multiplication by the scalar $z$.
- (a) Show that $Z\subset M_n(\mathbb{R})$ is a subspace.
- (b) Show that the centre of $M_n(\mathbb{R})$ consists of scalar matrices: $Z=\mathrm{Span}_{\mathbb{R}}(I_n)$.

4. Let $A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in M_2(\mathbb{R})$ and suppose that $ad-bc\neq 0$.
- (a) Find a matrix $B=\begin{pmatrix}e & f\\ g & h\end{pmatrix}$ such that $AB=I_2=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$. Show that $BA=I_2$ as well.
- (*b) ("Uniqueness of the inverse") Suppose that $AC=I_2$. Show that $C=B$.

*5. Find a matrix $N\in M_2(\mathbb{R})$ such that $N^2=0$ but $N\neq 0$.

6. ("Group homomorphisms")
- (a) Let $R_\alpha$ be the matrix $\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$ ("rotation in the plane by angle $\alpha$"). Show that $R_\alpha R_\beta=R_{\alpha+\beta}$.
- (b) Let $n(x)$ be the matrix $\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$ ("shear in the plane by $x$"). Show that $n(x)n(y)=n(x+y)$.

## An application to graph theory

*7. Let $V$ be a vector space. A linear map $T:V\to V$ is said to be *bipartite* if there are subspaces $W_1,W_2\subset V$ such that $V=W_1\oplus W_2$ (internal direct sum). and such that $T(W_1)\subset W_2$ and $T(W_2)\subset W_1$. Let $T$ be bipartite with respect to the decomposition $V=W_1\oplus W_2$. Show that $\dim\mathrm{Ker}\,T\geq|\dim W_1-\dim W_2|$.

Hint for 2: interpret the compositions as linear maps, and use the practice problem.
Hint for 3a: use the practice problem and a previous problem set.

## Supplementary problems

A. Show by hand that for any three matrices $A, B, C$ with compatible dimensions, $(AB)C = A(BC)$.

B. (Every vector space is $\mathbb{R}^n$) Let $V$ be a vector space with basis $B = \{\underline{v}_i\}_{i\in I}$ ($I$ may be infinite).

(a) Let $\Phi\colon \mathbb{R}^{\oplus I} \to V$ be the map $\Phi(f) = \sum_{i\in I} f_i \underline{v}_i = \sum_{f_i \neq 0} f_i \underline{v}_i$ [recall that we admit infinite sums where only finitely many summands are non zero]. Show that $\Phi$ is a an isomorphism of vector spaces.

RMK The inverse map $\Psi\colon V \to \mathbb{R}^{\oplus I}$ is called the *coordinate map* (in the ordered basis $B$)

(b) Construct an isomorphism $V^* \to \mathbb{R}^I$.

(c) Let $W$ be another space with basis $C = \left\{\underline{w}_j\right\}_{j\in J}$. Construct an injective linear map $\mathrm{Hom}(V,W) \to M_{I\times J}(\mathbb{R}) = \mathbb{R}^{I\times J}$ and show that its image is the set of matrices having at most finitely many non-zero entries in each column.