Lior Silberman's Math 223: Problem Set 3 (due 1/2/2021)

Practice problems (recommended, but do not submit)

Section 1.6, Problems 1 (except (g)), 2-5, 7, 11,12, 22*, 24*.

Bases and dimension

1. (§1.6 E8)Let $W = \left\{\underline{x} \in \mathbb{R}^5 \mid \sum_{i=1}^{5} x_i = 0\right\}$ be the set of vectors in $\mathbb{R}^5$ whose co-ordinates sum to zero. It is a subspace (but you don't have to check this). The following 8 vectors span W (you don't have to check that either). Find a subset of them which forms a basis for W. $\underline{u}_1 = (2,-3,4,-5,2)$, $\underline{u}_2 = (-6,9,-12,15,-6)$, $\underline{u}_3 = (3,-2,7,-9,1)$, $\underline{u}_4 = (2,-8,2,-2,6)$, $\underline{u}_5 = (-1,1,2,1,-3)$, $\underline{u}_6 = (0,-3,-18,9,12)$, $\underline{u}_7 = (1,0,-2,3,-2)$, $\underline{u}_8 = (2,-1,1,-9,7)$.

2. Find a basis for the subspace $\left\{\underline{x} \in \mathbb{R}^4 \mid x_1 + 3x_2 - x_3 = 0\right\}$ of $\mathbb{R}^4$. What is the dimension?

3. Let $U = \left\{p \in \mathbb{R}[x]^{\leq n} \mid p(-x) = p(x)\right\}$. Find a basis for U and determine its dimension.

*4. Let $\mathbb{R}(x)$ be the space of functions of the form $\frac{f}{g}$ where $f,g \in \mathbb{R}[x]$ are polynomials. $\mathbb{R}(x)$ is called "the field of rational functions in one variable, and has the same relation to the ring of polynomials $\mathbb{R}[x]$ that the rational numbers $\mathbb{Q}$ have to the ring of integers $\mathbb{Z}$. We will consider $\mathbb{R}(x)$ as a real vector space.
   (a) Show that $\frac{1}{1-x} \in \mathbb{R}(x)$ is linearly independent of the set $\left\{x^k\right\}_{k=0}^{\infty} \subset \mathbb{R}(x)$.
   RMK It's true that $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ holds on the interval $(-1,1)$, but don't forget that the summation symbol on the left *does not stand* for repeated addition. Rather, it stands for a kind of limit.
   (b) Show that the subset $\left\{\frac{1}{x-a}\right\}_{a\in\mathbb{R}} \subset \mathbb{R}(x)$ is linearly independent.
   RMK The vector space $\mathbb{R}[x]$ has countable dimension, but by part (b) the dimension of $\mathbb{R}(x)$ as a real vector space is at least the cardinality of the continuum. In fact there is equality, because the cardinality of all of $\mathbb{R}(x)$ is that of the continuum.

Linear Functionals

Fix a vector space V. A *linear functional* on V is a map $\varphi\colon V \to \mathbb{R}$ such that for all $a,b \in \mathbb{R}$ and $\underline{u},\underline{v} \in V$, $\varphi(a\underline{v} + b\underline{u}) = a\varphi(\underline{v}) + b\varphi(\underline{u})$. Let $V^* \stackrel{\text{def}}{=} \{\varphi\colon V \to \mathbb{R} \mid \varphi \text{ is a linear functional}\}$ be the set of linear functionals on V (called vector space *dual* to V, in short the *dual space*).

5. (The basic example)
   (a) Show that $\varphi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x - 2y + 3z$ defines a linear functional on $\mathbb{R}^3$.
   (b) Let φ be a linear functional on $\mathbb{R}^2$. Show that $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = x\cdot\varphi\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + y\cdot\varphi\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, thus that every linear functional on $\mathbb{R}^2$ is of the form $\varphi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = ax + by$ for some $a,b \in \mathbb{R}$.
   SUPP Construct an identification of $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$.

6. Show that V^* is a subspace of $\mathbb{R}^V$, hence a vector space.

7. Let V be a vector space and let $\varphi \in V^*$ be non-zero.
   (a) Show that $\operatorname{Ker}\varphi \stackrel{\text{def}}{=} \{\underline{v} \in V \mid \varphi(\underline{v}) = 0\}$ is a subspace.
   (*b) Show that there is $\underline{v} \in V$ satisfying $\varphi(\underline{v}) = 1$.
   (**c) Let B be a basis of $\operatorname{Ker}\varphi$, and let $\underline{v} \in V$ be as in part (b). Show that $B \cup \{\underline{v}\}$ is a basis of V.

RMK If V is finite-dimensional this shows: $\dim V = \dim \operatorname{Ker}\varphi + 1$. In general we say that $\operatorname{Ker}\varphi$ is of *codimension* 1.

A Linear Transformation

In this problem our choice of letters follows conventions from physics. Thus v will be a numerical parameter rather than a vector, and we write the coordinates of a vector in $\mathbb{R}^2$ as $\begin{pmatrix} x \\ t \end{pmatrix}$ rather than $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

8. In the course of his researches on electromagnetism, Henri Poincaré wrote down the following map $L_v\colon \mathbb{R}^2 \to \mathbb{R}^2$ which he called the "Lorentz transformation":

$$L_v \begin{pmatrix} x \\ t \end{pmatrix} \stackrel{\text{def}}{=} \gamma_v \cdot \begin{pmatrix} x - vt \\ t - vx \end{pmatrix} .$$

Here v is a real parameter such that $|v| < 1$ and γ_v is also a number, defined by $\gamma_v = \left(1 - v^2\right)^{-1/2}$.

(a) Suppose $v = 0.6$ so that $\gamma_v = (1 - 0.6^2)^{-1/2} = 1.25$. Calculate $L_v \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $L_v \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $L_v \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Check that $L_v \begin{pmatrix} 2 \\ 3 \end{pmatrix} = L_v \begin{pmatrix} 3 \\ 2 \end{pmatrix} + L_v \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

(b) Show that L_v is a linear transformation.

(c) ("Relativistic addition of velocities") Let $v, v' \in (-1,1)$ be two parameters. Show that $L_v \circ L_{v'} = L_u$ for $u = \frac{v+v'}{1+vv'}$. It is a fact that if $v, v' \in (-1,1)$ then $\frac{v+v'}{1+vv'} \in (-1,1)$ as well.
Hint: Start by showing $\gamma_v \gamma_{v'} = \frac{\gamma_u}{1+vv'}$.

RMK If $g\colon A \to B$ and $f\colon B \to C$ are functions then $f \circ g$ denotes their *composition*, the function $f \circ g\colon A \to C$ such that $(f \circ g)(a) = f(g(a))$ for all $a \in A$.

Supplementary problems

A. Let V be a vector space and let $W_1, W_2 \subset V$ be finite-dimensional subspaces.
  (a) Show that $\dim(W_1 + W_2) \leq \dim W_1 + \dim W_2$.
  (**b) Show that $\dim(W_1 + W_2) + \dim(W_1 \cap W_2) = \dim W_1 + \dim W_2$.
  RMK Let A, B be finite sets. Then the "inclusion-exclusion" formula states $\#A + \#B = \#(A \cup B) + \#(A \cap B)$

B. Let V be a vector space, W a subspace. Let $B \subset W$ be a basis for W and let $C \subset V$ be disjoint from B and such that $B \cup C$ is a basis for V (that is, we extend B until we get a basis for V).
  (a) Show that $\{\underline{v} + W\}_{\underline{v} \in C}$ is a basis for the quotient vector space V/W (V/W is defined in the supplement to PS2).
  (b) Show that $\dim W + \dim(V/W) = \dim V$.

The following problem requires some background in set theory.

C. Let V be a vector space, and let B, C be a bases of V.
  (a) Suppose one of B, C is finite, Show that the other is finite and that they have the same size.
  – We may therefore assume both B, C are finitely.
  (b) For a finite subset $A \subset B$ show that $C \cap \mathrm{Span}(A)$ is finite.
  – Let $\mathcal{F}_B, \mathcal{F}_C$ be the sets of finite subsets of B, C respectively, and let $f \colon \mathcal{F}_B \to \mathcal{F}_C$ be the function $f(A) = C \cap \mathrm{Span}(A)$.
  (c) Show that the image of f covers C.
  (d) Show that the cardinality of the image of f is at least that of C.
  (e) Show that $|B| \geq |C|$. Conclude that $|B| = |C|$, in other words that infinite-dimensional vector spaces also have well-defined dimensions.