

Lior Silberman's Math 412: Problem set 10, not for submission

The exponential

1. Products of absolutely convergent series.

(a) Let V be a normed space, and let $T, S \in \text{End}_b(V)$ commute. Show that $\exp(T + S) = \exp(T)\exp(S)$.

(b) Show that, for appropriate values of t , $\exp(A)\exp(B) \neq \exp(A + B)$ where $A = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$,

$$B = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}.$$

Companion matrices

DEF The *companion matrix* associated with the polynomial $p(x) = x^n - \sum_{i=0}^{n-1} a_i x^i$ is

$$C = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}.$$

2. A sequence $\{x_k\}_{k=0}^{\infty}$ is said to satisfy a *linear recurrence relation* if for each k ,

$$x_{k+n} = \sum_{i=0}^{n-1} a_i x_{k+i}.$$

(a) Define vectors $\underline{v}^{(k)} = (x_{k-n+1}, x_{k-n+2}, \dots, x_k)$. Show that $\underline{v}^{(k+1)} = C\underline{v}^{(k)}$ where C is the companion matrix.

(b) Find x_{100} if $x_0 = 1, x_1 = 2, x_2 = 3$ and $x_n = x_{n-1} + x_{n-2} - x_{n-3}$.

PRAC Find the Jordan canonical form of $\begin{pmatrix} 1 & & \\ & 1 & \\ 0 & 0 & 2 \end{pmatrix}$.

3. Let C be the companion matrix associated with the polynomial $p(x) = x^n - \sum_{k=0}^{n-1} a_k x^k$.

(a) Show that $p(x)$ is the characteristic polynomial of C .

(b) Show that $p(x)$ is also the minimal polynomial.

– For parts (c),(d) fix a non-zero root λ of $p(x)$.

(c) Find (with proof) an eigenvector with eigenvalue λ .

(**d) Let g be a polynomial, and let \underline{v} be the vector with entries $v_k = \lambda^k g(k)$ for $0 \leq k \leq n-1$.

Show that, if the degree of g is small enough (depending on p, λ), then $((C - \lambda)\underline{v})_k = \lambda(g(k+1) - g(k))\lambda^k$ and (the hard part) that

$$((C - \lambda)\underline{v})_{n-1} = \lambda(g(n) - g(n-1))\lambda^{n-1}.$$

(**e) Find the Jordan canonical form of C .

Holomorphic calculus

Let $f(z) = \sum_{m=0}^{\infty} a_m z^m$ be a power series with radius of convergence R . For a matrix A define $f(A) = \sum_{m=0}^{\infty} a_m A^m$ if the series converges absolutely in some matrix norm.

4. Let $D = \text{diag}(\lambda_1, \dots, \lambda_n)$ be diagonal with $\rho(D) < R$ (that is, $|\lambda_i| < R$ for each i). Show that $f(D) = \text{diag}(f(\lambda_1), \dots, f(\lambda_n))$.
5. Let $A \in M_n(\mathbb{C})$ be a matrix with $\rho(A) < R$.
 - (a) [review of power series] Let R' satisfy $\rho(A) < R' < R$. Show that $|a_m| \leq C(R')^{-m}$ for some $C > 0$.
 - (b) Using PS8 problem 3(a) show that $f(A)$ converges absolutely with respect to any matrix norm.
 - (*c) Suppose that $A = S(D+N)S^{-1}$ where $D+N$ is the Jordan form (D is diagonal, N upper-triangular nilpotent). Show that

$$f(A) = S \left(\sum_{k=0}^n \frac{f^{(k)}(D)}{k!} N^k \right) S^{-1}.$$

Hint: D, N commute.

RMK1 This gives an alternative proof that $f(A)$ converges absolutely if $\rho(A) < R$, using the fact that $f^{(k)}(D)$ can be analyzed using single-variable methods.

RMK2 Compare your answer with the Taylor expansion $f(x+y) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)}{k!} y^k$.

- (d) Apply this formula to find $\exp(tB)$ where B is as in PS9 problem 2.
6. Let $A \in M_n(\mathbb{C})$. Prove that $\det(\exp(A)) = \exp(\text{Tr}A)$.

Supplementary problems

- A. Let $p \in \mathbb{C}[x]$ be a polynomial, let D' be the derivative operator for distributions in $C_c^\infty(\mathbb{R})'$. Show that $\varphi \in C_c^\infty(\mathbb{R})'$ satisfies $p(D')\varphi = 0$ iff φ is given by integration against a function f such that $p(D)f = 0$.