Lior Silberman's Math 412: Problem Set 3 (due 26/9/2019)

Practice

P1 Let $\underline{u}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$, $\underline{u}_2 = \begin{pmatrix}1\\1\\-1\end{pmatrix}$, $\underline{u}_3 = \begin{pmatrix}1\\-1\\-1\end{pmatrix}$, $\underline{u} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ as vectors in $\mathbb{R}^3$.

(a) Construct an explicit linear functional $\varphi \in \left(\mathbb{R}^3\right)'$ vanishing on $\underline{u}_1, \underline{u}_2$.
(b) Show that $\{\underline{u}_1, \underline{u}_2, \underline{u}_3\}$ is a basis on $\mathbb{R}^3$ and find its dual basis.
(c) Evaluate the dual basis at $\underline{u}$.

P2 Let V be n-dimensional and let $\{\varphi_i\}_{i=1}^m \in V'$.

(a) Show that if $m < n$ there is a non-zero $\underline{v} \in V$ such that $\varphi_i(\underline{v}) = 0$ for all i. Interpret this as a statement about linear equations.
(b) When is it true that for each $\underline{x} \in F^m$ there is $\underline{v} \in V$ such that for all i, $\varphi_i(\underline{v}) = x_i$?

P3 Let U, V be finite-dimensional vector spaces and let $L \in \mathrm{Hom}_F(U,V)$. Consider the pairing $V' \times U \to F$ given by $\langle \varphi, \underline{u} \rangle_L = \varphi(L\underline{u})$. Let $\{\underline{u}_j\} \subset U$, $\{\underline{v}_i\} \subset V$ be bases and let $\{\varphi_i\} \subset V'$ be the basis dual to $\{\underline{v}_i\}$. Show that the matrix of L as a linear map $U \to V$ is the same as the Gram matrix of the pairing $\langle \cdot, \cdot \rangle_L$.

Example of linear functionals: Banach limits

Recall that $\ell^\infty \subset \mathbb{R}^{\mathbb{N}}$denote the set of *bounded* sequences (the sequences $\underline{a}$ such that for some M we have $|a_i| \le M$ for all i). Let $S\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ be the *shift* map $(S\underline{a})_n = \underline{a}_{n+1}$. A subspace $U \subset \mathbb{R}^{\mathbb{N}}$ is *shift-invariant* if $S(U) \subset U$. If U is shift-invariant a function F with domain U is called *shift-invariant* if $F \circ S = F$ (example: the subset $c \subset \mathbb{R}^{\mathbb{N}}$ of convergent sequences is a shift-invariant subspace, as is the functional $\lim\colon c \to \mathbb{R}$ assigning to every sequence its limit).

Note that P4 is a practice problem!

P4 (Useful facts)

(a) Show that ℓ^∞ is a subspace of $\mathbb{R}^{\mathbb{N}}$.
(b) Show that $S\colon \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is linear and that $S(\ell^\infty) = \ell^\infty$.
(c) Let $U \subset \mathbb{R}^{\mathbb{N}}$ be a shift-invariant subspace. Show that the set $U_0 = \{S\underline{a} - \underline{a} \mid \underline{a} \in U\}$ is a subspace of U.
(d) In the case $U = \mathbb{R}^{\oplus\mathbb{N}}$ of sequences of finite support, show that $U_0 = U$.
(e) Let Z be an auxiliary vector space. Show that $F \in \mathrm{Hom}(U,Z)$ is shift-invariant iff F vanishes on U_0.

1. Let $W = \{S\underline{a} - \underline{a} \mid \underline{a} \in \ell^\infty\} \subset \ell^\infty$. Let $\mathbb{1}$ be the sequences everywhere equal to 1.

(a) Show that the sum $W + \mathbb{R}\mathbb{1} \subset \ell^\infty$ is direct and construct an S-invariant functional $\varphi\colon \ell^\infty \to \mathbb{R}$ such that $\varphi(\mathbb{1}) = 1$ (*Hint*: PS2 problem 5(b)).
(b) (Strengthening) For $\underline{a} \in \ell^\infty$ set $\|\underline{a}\|_\infty = \sup_n |a_n|$. Show that if $\underline{a} \in W$ and $x \in \mathbb{R}$ then $\|\underline{a} + x\mathbb{1}\|_\infty \ge |x|$. (Hint: consider the average of the first N entries of the vector $\underline{a} + x\mathbb{1}$).

SUPP Let $\varphi \in (\ell^\infty)'$ be shift-invariant, positive (if $a_i \ge 0$ for all i then $\varphi(\underline{a}) \ge 0$), and satisfy $\varphi(\mathbb{1}) = 1$. Show that $\liminf_{n\to\infty} a_n \le \varphi(\underline{a}) \le \limsup_{n\to\infty} a_n$ and conclude that the restriction of φ to c is the usual limit.

2. ("choose one") Let $\varphi \in (\ell^\infty)'$ satisfy $\varphi(\mathbb{1}) = 1$. Let $\underline{a}$ be the sequence $a_n = \frac{1+(-1)^n}{2}$.
   (a) Suppose that φ is shift-invariant. Show that $\varphi(\underline{a}) = \frac{1}{2}$.
   (b) Suppose that φ respects pointwise multiplication (if $z_n = x_n y_n$ then $\varphi(\underline{z}) = \varphi(\underline{x})\varphi(\underline{y})$). Show that $\varphi(\underline{a}) \in \{0,1\}$.

Duality and bilinear forms

3. (The dual map) Let U,V,W be vector spaces, and let $T \in \mathrm{Hom}(U,V)$, and let $S \in \mathrm{Hom}(V,W)$.
   (a) (The abstract meaning of transpose) Suppose U,V be finite-dimensional with bases $\{\underline{u}_j\}_{j=1}^m \subset U$, $\{\underline{v}_i\}_{i=1}^n \subset V$, and let $A \in M_{n,m}(F)$ be the matrix of T in those bases. Show that the matrix of the dual map $T' \in \mathrm{Hom}(V',U')$ with respect to the dual bases $\left\{\underline{u}'_j\right\}_{j=1}^m \subset U'$, $\{\underline{v}'_i\}_{i=1}^n \subset V'$ is the transpose ${}^{\mathrm{t}}A$.
   (b) Show that $(ST)' = T'S'$. It follows that ${}^{\mathrm{t}}(AB) = {}^{\mathrm{t}}B\,{}^{\mathrm{t}}A$.

4. Let $F^{\oplus\mathbb{N}}$ denote the space of sequences of finite support. Construct a non-degenerate pairing $F^{\oplus\mathbb{N}} \times F^{\mathbb{N}} \to F$, giving a concrete realization of $\left(F^{\oplus\mathbb{N}}\right)'$.

5. Let $C_c^\infty(\mathbb{R})$ be the space of compactly supported smooth functions on $\mathbb{R}$ (that is, functions which have derivatives of all orders and which are identically zero outside some interval), and let $D\colon C_c^\infty(\mathbb{R}) \to C_c^\infty(\mathbb{R})$ be the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}x}$. For a reasonable function f on $\mathbb{R}$ define a functional φ_f on $C_c^\infty(\mathbb{R})$ by $\varphi_f(g) = \int_{\mathbb{R}} f g \,\mathrm{d}x$ (note that f need only be integrable, not continuous).
   (a) Show that if f is continuously differentiable then $D'\varphi_f = \varphi_{-Df}$. (*Hint*: this expresses a basic fact from calculus)
   DEF For this reason one usually extends the operator D to the dual space by $D\varphi \overset{\mathrm{def}}{=} -D'\varphi$, thus giving a notion of a "derivative" for non-differentiable and even discontinuous functions.
   (b) Let the "Dirac delta" $\delta \in C_c^\infty(\mathbb{R})'$ be the evaluation functional $\delta(f) = f(0)$. Express $(D\delta)(f)$ in terms of f.
   (c) Let φ be a linear functional such that $D'\varphi = 0$. Show that for some constant c, $\varphi = \varphi_{c\mathbb{1}}$.

Supplement: The support of distributions

A. (This is a mostly a problem in analysis) Let $\varphi \in C_c^\infty(\mathbb{R})'$.
   DEF Let $U \subset \mathbb{R}$ be open. Say that φ is *supported away from* U if for any $f \in C_c^\infty(U)$, $\varphi(f) = 0$. The *support* $\mathrm{supp}(\varphi)$ is the complement the union of all such U.
   (a) Show that $\mathrm{supp}(\varphi)$ is closed, and that φ is supported away from $\mathbb{R} \setminus \mathrm{supp}(\varphi)$.
   (b) Show that $\mathrm{supp}(\delta) = \{0\}$ (see problem 5(b)).
   (c) Show that $\mathrm{supp}(D\varphi) \subset \mathrm{supp}(\varphi)$ (note that this is well-known for functions).
   (d) Show that $D\delta$ is not of the form φ_f for any function f.