Lior Silberman's Math 412: Problem set 10, due 23/11/2017

The exponential

1. Products of absolutely convergent series.
   (a) Let V be a normed space, and let $T,S \in \mathrm{End}_{\mathrm{b}}(V)$ commute. Show that $\exp(T+S) = \exp(T)\exp(S)$.
   (b) Show that, for appropriate values of t, $\exp(A)\exp(B) \neq \exp(A+B)$ where $A = \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ -t & 0 \end{pmatrix}$.

Companion matrices

DEF The *companion matrix* associated with the polynomial $p(x) = x^n - \sum_{i=0}^{n-1} a_i x^i$ is

$$C = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}.$$

2. A sequence $\{x_k\}_{k=0}^{\infty}$ is said to satisfy a *linear recurrence relation* if for each k,

$$x_{k+n} = \sum_{i=0}^{n-1} a_i x_{k+i}.$$

   (a) Define vectors $\underline{v}^{(k)} = (x_{k-n+1}, x_{k-n+2}, \ldots, x_k)$. Show that $\underline{v}^{(k+1)} = C\underline{v}^{(k)}$ where C is the companion matrix.

   (b) Find x_{100} if $x_0 = 1$, $x_1 = 2$, $x_2 = 3$ and $x_n = x_{n-1} + x_{n-2} - x_{n-3}$.

PRAC Find the Jordan canonical form of $\begin{pmatrix} & 1 & \\ & & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

3. Let C be the companion matrix associated with the polynomial $p(x) = x^n - \sum_{k=0}^{n-1} a_k x^k$.
   (a) Show that $p(x)$ is the characteristic polynomial of C.
   (b) Show that $p(x)$ is also the minimal polynomial.
   – For parts (c),(d) fix a non-zero root λ of $p(x)$.
   (c) Find (with proof) an eigenvector with eigenvalue λ.
   (**d) Let g be a polynomial, and let $\underline{v}$ be the vector with entries $v_k = \lambda^k g(k)$ for $0 \leq k \leq n-1$. Show that, if the degree of g is small enough (depending on p, λ), then $((C-\lambda)\underline{v})_k = \lambda\,(g(k+1) - g(k))\,\lambda^k$ and (the hard part) that

$$((C-\lambda)\underline{v})_{n-1} = \lambda\,(g(n) - g(n-1))\,\lambda^{n-1}.$$

   (**e) Find the Jordan canonical form of C.

Holomorphic calculus

Let $f(z)=\sum_{m=0}^{\infty}a_m z^m$ be a power series with radius of convergence R. For a matrix A define $f(A)=\sum_{m=0}^{\infty}a_m A^m$ if the series converges absolutely in some matrix norm.

5. Let $D=\operatorname{diag}(\lambda_1,\cdots,\lambda_n)$ be diagonal with $\rho(D)<R$ (that is, $|\lambda_i|<R$ for each i). Show that $f(D)=\operatorname{diag}(f(\lambda_1),\cdots,f(\lambda_n))$.

6. Let $A\in M_n(\mathbb{C})$ be a matrix with $\rho(A)<R$.
   (a) [review of power series] Let R' satisfy $\rho(A)<R'<R$. Show that $|a_m|\leq C(R')^{-m}$ for some $C>0$.
   (b) Using PS8 problem 3(a) show that $f(A)$ converges absolutely with respect to any matrix norm.
   (*c) Suppose that $A=S(D+N)S^{-1}$ where $D+N$ is the Jordan form (D is diagonal, N upper-triangular nilpotent). Show that
   $$f(A)=S\left(\sum_{k=0}^{n}\frac{f^{(k)}(D)}{k!}N^k\right)S^{-1}.$$
   Hint: D,N commute.
   RMK1 This gives an alternative proof that $f(A)$ converges absolutely if $\rho(A)<R$, using the fact that $f^{(k)}(D)$ can be analyzed using single-variable methods.
   RMK2 Compare your answer with the Taylor expansion $f(x+y)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x)}{k!}y^k$.
   (d) Apply this formula to find $\exp(tB)$ where B is as in PS9 problem 2.

7. Let $A\in M_n(\mathbb{C})$. Prove that $\det(\exp(A))=\exp(\operatorname{Tr}A)$.

Supplementary problems

A. Let $p\in\mathbb{C}[x]$ be a polynomial, let D' be the derivative operator for distributions in $C_c^\infty(\mathbb{R})'$. Show that $\varphi\in C_c^\infty(\mathbb{R})'$ satisfies $p(D')\varphi=0$ iff φ is given by integration against a function f such that $p(D)f=0$.