Math 322, lecture 19, 14/11/2017

Note: PS9 due Tue 21/11
PS10 due Tue 28/11

Last time: applications of Sylow thms:
- Classification of groups of order 12
- $\#G = 30 \Rightarrow G$ not simple ("element counting")

Example: $G = SL_2(\mathbb{R}) = \{ g \in M_2(\mathbb{R}) \mid \det g = 1 \}$

$\mathbb{H} = \{ x+iy \mid y>0 \}$. For $z \in \mathbb{H}$, $g \in G$ set

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\qquad g \cdot z = \dfrac{az+b}{cz+d}$

Ex: (1) $\operatorname{Im}(gz) = \dfrac{\operatorname{Im}(z)}{|cz+d|^2} > 0$ $\qquad$ (2) $h \cdot g \cdot z = (hg) \cdot z$

Note: (①) $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \cdot z = z + x$, $\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \cdot z = y \cdot z$

$\Rightarrow \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \cdot z = x + yz$

$\Rightarrow \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & \\ & 1/\sqrt{y} \end{pmatrix} \cdot i = x + iy$

i.e. action is <u>transitive</u>: only one orbit.

(and the subgp $NA \cong N \rtimes A$ acts simply transitively:
stabilizer is trivial)

More: Set $K = Stab_G(i) = \{g \in SL_2(\mathbb{R}) \mid g \cdot i = i\}$

Ex: $K = SO(2)$ ← 2x2 orthogonal matrices, det = 1

Claim: Map $N \times A \times K \to G$ given by $(n, a, k) \mapsto nak$ is a bijection

Pf: Let $g \in G$. Then $g \cdot i \in \mathbb{H}$, and NA acts transitively, so have $na \in NA$ s.t. $g \cdot i = (na) \cdot i$

so $(na)^{-1} \cdot g \cdot i = i$

so $k = (na)^{-1} g \in K$ then $nak = g$

Suppose $g = nak = n'a'k'$ then $(nak) \cdot i = (n'a'k') \cdot i$

but $ki = k'i = i$, so $(na) \cdot i = (n'a') \cdot i$

But NA acts simply transitively so $na = n'a'$

so $g = nak = nak'$ so $k = k'$.

Back to Sylow thms

Example: Let G be a simple group of order 60. Then $G \cong A_5$.

Pf: $60 = 2^2 \cdot 3 \cdot 5$.

$n_2(G) \in \{1, 3, 5, 15\}$

$n_3(G) \in \{1, ~~2~~, 4, ~~5~~, 10, ~~20~~\}$

$n_5(G) \in \{1, ~~2~~, ~~4~~, ~~3~~, 6, ~~12~~\}$

Suppose $2 \le n_p(G) \le 4$ for $p | 60$. Then action of G on $Syl_p(G)$ gives a non-trivial hom $G \to S_{n_p(G)}$ (image is a transitive subgp)

~~[illegible]~~ But ~~[illegible]~~ $\# S_{n_p(G)} \le 24 < 60$ in this case, so kernel of this hom would be a non-trivial normal subgp ($\ne G$), a contradiction

Can't have $n_p(G) = 1$ - then P_p would be normal.

Conclusion: $n_2(G) \in \{5, 15\}$, $n_3(G) \in \{10\}$, $n_5(G) \in \{6\}$

Case 1: $n_2(G) = 5$. Then the action of G on $Syl_2(G)$ gives a non-trivial hom $G \to S_5$ (necc. an injection) image is a subgp of S_5 of order 60. Why is it A_5?

G simple $\Rightarrow$ $\ker f = \{e\}$ $\Rightarrow$ f injective

Pf 1: Let $H < S_n$ have index 2. Then $H \trianglelefteq S_n$. $(n \ge 5)$ (either since 2 is smallest prime dividing $n!$ or by hand) in particular, any subgp of S_5 of order 60.

HW: A_n only normal subgp other than $\{e\}$, S_n.

Pf 2: G has 10 3-sylow subgps, each $\cong C_3$. They are thus disjoint, so G has 20 elements of order 3

A_5 is simple of order 60, so A_5 has 20 such too. (and all 3-cycles in S_5 are in A_5) so image of hom contains all 3-cycles in S_5, hence it contains A_5

Case 2: $n_2(G) = 15$.

G has 20 elements of order 3

G has $6 \cdot 4 = 24$ " " " 5 (5-Sylow subgps are $\cong C_5$, hence disjoint)

$\Rightarrow$ G has at most $60 - 20 - 24 - 1 = 15$ elements of order 2 or 4.

So 2-Sylow subgps ~~are~~ cannot be disjoint.

So let $x \in G$ lie in <u>two</u> 2-Sylow subgps

Consider $C_G(x) = \{g \in G \mid gxg^{-1} = x\}$

Now $\# P_2 = 4$ so P_2 is commutative.

So $C_G(x)$ contains two distinct 2-Sylow subgps

Now $[G : P_2] = \frac{60}{4} = 15$, and $C_G(x) \supsetneq P_2$

So $[G : C_G(x)]$ is a proper divisor of 15, i.e.

~~the exact~~ one of 1, 3, 5.

$G \neq C_G(x)$ because $Z(G) = \{e\}$ (G is simple)

So $[G : C_G(x)] \neq 1$.

action $g : xC_G(x) = (gx) C_G(x)$

Now let G act <u>by translation</u> on $G / C_G(x)$

This is transitive, hence non-trivial. But G can't act on a set of size 3 non-trivially (can't embed $G \hookrightarrow S_3$)

So $[G : C_G(x)] = 5$ and G has a hom $\to S_5$. ∎

To see the image is A_5, proceed as in Case 1.

Remarks We repeatedly used:

(1) If G is simple, $n_p(G) \neq 1$ (no normal sylow subgp)

(2) If G is simple, $f \in \mathrm{Hom}(G, H)$ either $\mathrm{Ker}(f) = G$ (f trivial) or $\mathrm{Ker}(f) = \{e_G\}$ (f an injection)

(3) If G is simple, $\#H < \#G$, then $\mathrm{Hom}(G, H) = \{\text{triv}\}$

Aside: let $\tilde{G} = SL_2(\mathbb{F}_5)$ ($\mathbb{F}_5 = \mathbb{Z}/5\mathbb{Z}$ is the field with 5 elements)

How many elements in $\tilde{G}$?

$\tilde{G}$ acts transitively on $\mathbb{F}_5^2 \setminus \{0\}$

$$\mathrm{Stab}_{\tilde{G}}\left(\begin{pmatrix}0\\1\end{pmatrix}\right) = \left\{ \begin{pmatrix}a & b\\ c & d\end{pmatrix} : \begin{pmatrix}b\\d\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \right\} = \left\{ \begin{pmatrix}a & 0\\ c & 1\end{pmatrix} \right\} \underset{\uparrow \det=1}{=} \left\{ \begin{pmatrix}1 & \\ c & 1\end{pmatrix} \right\}$$

($\uparrow$ gp of order 5)

So $\dfrac{\#\tilde{G}}{\#\mathrm{Stab}} = \#(\mathbb{F}_5^2 \setminus \{0\})$ ← orbit-stabilizer thm $\Rightarrow \#\tilde{G} = (24) \cdot 5 = 120$

Notes $\tilde{G} \not\cong S_5$: $\begin{pmatrix}-1 & \\ & -1\end{pmatrix} \in Z(\tilde{G})$

in fact, $Z(\tilde{G}) = \left\{ \begin{pmatrix}1 & \\ & 1\end{pmatrix}, \begin{pmatrix}-1 & \\ & -1\end{pmatrix} \right\}$

Then $G = \tilde{G}/Z(\tilde{G})$ has order 60 (G is called $PSL_2(\mathbb{F}_5)$)

Fact: G is simple.

Conclusion: $G \cong A_5$.