Math 322, lecture 18, 9/11/2017

Last time:

Thm: (Sylow) Let G be a finite group of order n, p prime s.t. $p^k \| n$. Then

(1) Every p-subgp of G is contained in a subgp of order p^k ("Sylow p-subgp")

(2) All Sylow p-subgps are conjugate

(3) $n_p(G) = \# \mathrm{Syl}_p(G)$ satisfies $n_p(G) | n$, $n_p(G) \equiv 1 \ (p)$

Remark: If $n = p^k m$, with $p \nmid m$ then (3) says:

$$n_p(G) \mid m, \quad n_p(G) \equiv 1 \ (p) \qquad (*)$$

Basic exercise: ("numerology") given n, list all prime p dividing n, for each p all solutions to $(*)$

Example: the groups of order 12 are:

$$C_{12},\ C_3 \times C_2 \times C_2 = C_6 \times C_2,\ A_4,\ C_2 \times S_3,\ C_4 \ltimes C_3$$

(Q: which of those is D_{12}?)

implicit: (1) these are pairwise distinct
(2) there is a unique (up to isom) non-commuting pdt $C_4 \ltimes C_3$.

Pf: $3 | 12$. $n_3(G)$ is a divisor of 4, and be $\equiv 1 \ (3)$ so $n_3(G) \in \{1, 4\}$

also $n_2(G) | 3$, $n_2(G) \equiv 1 \ (2)$ so $n_2(G) \in \{1, 3\}$

<u>Case 1:</u> $n_3(G) = 4$. We know G acts by conjugation on $Syl_3(G)$. This is an action on a set of size 4, hence a hom $f: G \to S_4$.

Let $P_3 \in Syl_3(G)$. Then $[G : N_G(P_3)] = \#$ of conjugates of P_3 $= 4$

$\#P_3 = 3$

$$\left.\begin{array}{rl} & G \\ 4 \to & | \\ & N_G(P_3) \\ & | \\ & P_3 \\ 3 \to & | \\ & \{e\} \end{array}\right) 12$$

But $[G : P_3] = \dfrac{\#G}{\#P_3} = \dfrac{12}{3} = 4$ so

$$[N_G(P_3) : P_3] = \frac{4}{4} = 1$$

so P_3 is its own normalizer.

$Ker(f) < P_3$ (the kernel of action stabilizes every point)

P_3 is not normal (it has 4 conjugates)

so $Ker(f) \neq P_3$, so ($P_3 \simeq C_3$ has no non-triv subgps)

~~P_3 is not normal but~~ $Ker(f) = \{e\}$

<u>Conclusion:</u> f is an isom onto its image: $G \simeq$ subgp of S_4 of order 12.

Now the different 3-Sylow subgps are disjoint:

if $P_3, P_3' \in Syl_3(G)$, $P_3 \cap P_3' < P_3$ if this was P_3, we'd get $P_3 \subset P_3'$, so $P_3 = P_3'$.

so instead $P_3 \cap P_3' = \{e\}$ (C_3 only has itself and $\{e\}$ for subgroups)

Each $P_3 \simeq C_3$ has two ~~elem~~ non-identity elements, so G has $8 = 4 \cdot 2$ elements of order 3

So $f(G) \leq S_4$ has 8 elements of order 3.

Only elements of S_4 of order 3 are 3-cycles, and there are 8 of them (4 ways to choose support, 2 cyclic orderings of $(1,2,3)$).

So the image of f contains all 3-cycles, hence the subgroup they generate: $f(G) \supseteq A_4$.

But $\# f(G) = \# G = 12 = \# A_4$ so $f: G \to A_4$ is an isomorphism.

<u>Conclusion:</u> If $n_3(G) = 4$ then $n_2(G) = n_2(A_4) = 3$
$$P_2 \cong C_2 \times C_2$$

<u>Case 2:</u> $n_3(G) = 1$. Now $P_3 \trianglelefteq G$.

Let $P_2 \in Syl_2(G)$. Then $\# P_2 = 2^2 = 4$, so $P_2 \cap P_3 = \{e\}$ (2^2 and 3 have no common divisors), and $\# P_2 P_3 = \# P_2 \cdot \# P_3 = 4 \cdot 3 = 12$

So $G = P_2 P_3$ where $P_2 \cap P_3 = \{e\}$, $P_3 \trianglelefteq G$, i.e. G is a semidirect pdt of a gp of order 4 and a gp of order 3.

It remains to classify actions of P_2 on P_3.

<u>Case 2a:</u> the action is trivial, P_2, P_3 commute:
$$G \cong P_2 \times P_3, \text{ i.e. } G \cong C_4 \times C_3 \cong C_{12}$$
or
$$G \cong (C_2 \times C_2) \times C_3 \cong C_2 \times C_6,$$
(and they are distinct since they have non-isom P_2's.

Case 2b: The action is non-trivial and $P_2 \cong V \cong C_2 \times C_2$

The action is a hom $\varphi : P_2 \to Aut(P_3) \cong Aut(C_3) \cong C_2$

φ is non-triv, so surjective (C_2 has only 2 subgps)

~~so~~ let $K = Ker(\varphi)$. Then $V/K \cong C_2$ (1^{st} isom thm)

so $\# K = 2$. Then $K \cong C_2$ and taking any $\ell \in V \setminus K$, the group $L = \{1, \ell\}$ is complementary: $V = K \times L$. (and $\varphi(\ell) =$ non-triv element of $Aut(C_3)$)

so $G \cong V \ltimes C_3 \cong K \times (L \ltimes C_3) \cong C_2 \times D_6 \cong C_2 \times S_3$.

Case 2c: The action is non-trivial and $P_2 \cong C_4$.

Say ~~a~~ $[1] \in \mathbb{Z}/4\mathbb{Z} \cong C_4$ is a generator, $\varphi : P_2 \to Aut(C_3) \cong C_2$ the action.

Then φ is determined by $\varphi([1])$, and if φ is non-trivial, must have $\varphi([1]) =$ non-triv element of C_2

~~$\Rightarrow$~~ Because 2 (order of this element) divides 4 (order of generator of C_4) such φ exists (see PS8).

So get a unique non-commuting semidirect product

$$C_4 \ltimes C_3.$$

Concretely this means: let $a \in C_4$ be the generator, then if $h \in C_3$: $a^i h a^{-i} = \begin{cases} h^{-1} & i \text{ odd} \\ h & i \text{ even} \end{cases}$

Fact: $\mathrm{Aut}(C_p) \cong (\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z} \cong C_{p-1}$. if p is prime

(if p is odd, $\mathrm{Aut}(C_{p^k}) \cong C_{p^k - p^{k-1}}$)

Example: There is no simple group of order 30.

Pf: Suppose G was a simple group of order 30.

Numerology gives $n_3(G) \in \{1, \cancel{2}, \cancel{5}, 10\} = \{1, 10\}$

divisors of 10

not $\equiv 1\ (3)$

$n_5(G) \in \{1, 6\}$.

But G is simple so P_3, P_5 not normal, so $n_3(G) = 10$
$n_5(G) = 6$

The 3- and 5-sylow subgps are cyclic of order 3, 5 respectively $(30 = 2 \cdot 3 \cdot 5)$, so the P_3's and P_5's are separately disjoint

So G has $10 \cdot 2 = 20$ elements of order 3
and $6 \cdot 4 = 24$ " " " 5.

But G doesn't have $44 > 30$ distinct elements, so G doesn't exist.