Lior Silberman's Math 322: Problem Set 6 (due 26/10/2017)

Practice problems

P1. Let G be a group and let X be a set of size at least 2. Fix $x_0 \in X$ and for $g \in G, x \in X$ set $g \cdot x = x_0$.

(a) Show that this operation satisfies $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g,h \in G, x \in X$.

(b) This is not a group action. Why?

P2. Let G act on X. Say that $A \subset X$ is *G-invariant* if for every $g \in G$, $a \in A$ we have $g \cdot a \in A$.

(a) Show that A is G-invariant iff $g \cdot A = A$ ($g \cdot A$ in the sense of problem 4(a)).

(b) Suppose A is G-invariant. Show that the restriction of the action to A (formally, the binary operation $\cdot \upharpoonright_{G \times A}$) is an action of G on A.

Simplicity of A_n

1. Let $V = \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$. Show that $V \lhd S_4$, so that S_4 is not simple.

2. (The normal subgroups of S_n) Let $N \lhd S_n$ with $n \geq 5$.

(a) Let G be a group and let $H \lhd G$ be a normal subgroup isomorphic to C_2. Show that $H < Z(G)$.

(b) Suppose that $N \cap A_n \neq \{\mathrm{id}\}$. Show that $N \supset A_n$ and conclude that $N = A_n$ or $N = S_n$.

(c) Suppose that $N \cap A_n = \{\mathrm{id}\}$. Show that N is isomorphic to a subgroup of C_2.

(d) Show that if $n \geq 3$ then $Z(S_n) = \{\mathrm{id}\}$, and conclude that in case (c) we must have $N = \mathrm{id}$.

3. Let X be an infinite set.

(a) Show that $S_X^{\mathrm{fin}} = \{\sigma \in S_X \mid \mathrm{supp}(\sigma) \text{ is finite}\}$ is a subgroup of S_X.

PRAC For finite $F \subset X$ there is a natural inclusion $S_F \hookrightarrow S_X$, which is a group homomorphism and an isomorphism onto its image. Let $\mathrm{sgn}_F \colon S_F \to \{\pm 1\}$ be the sign character.

DEF For $\sigma \in S_X^{\mathrm{fin}}$ define $\mathrm{sgn}(\sigma) = \mathrm{sgn}_F(\sigma)$ for any finite F such that $\sigma \in S_F$.

(c) Show that $\mathrm{sgn}(\sigma)$ is well-defined (independent of F) and a group hom $S_X^{\mathrm{fin}} \to \{\pm 1\}$.

(*d) The *infinite alternating group* A_X is kernel of this homomorphism. Show that A_X is simple.

Group actions

4. Let the group G act on the set X.

(a) For $g \in G$ and $A \in P(X)$ set $g \cdot A = \{g \cdot a \mid a \in A\} = \{x \in X \mid \exists a \in A : x = g \cdot a\}$. Show that this defines an action of G on $P(X)$.

(b) In PS2 we endowed $P(X)$ with a group structure. Show that the action of (a) is by *automorphisms*: that the map $A \mapsto g \cdot A$ is a group homomorphism $(P(X), \Delta) \to (P(X), \Delta)$.

(c) Let Y be another set. For $f \colon X \to Y$ set $(g \cdot f)(x) = f(g^{-1} \cdot x)$. Show that this defines an action of G on Y^X, the set of functions from X to Y.

(*d) Suppose that $Y = \mathbb{R}$ (or any field), so that $\mathbb{R}^X$ has the structure of a vector space over $\mathbb{R}$. Show that the action of (c) is by *linear maps.*

5. (Some stabilizers) The action of S_X on X induces an action on $P(X)$ as in problem 4(a). Suppose that X is finite, $\#X = n$.
   (a) Let $A, B \subset X$. Show there is $\sigma \in S_X$ such that $\sigma \cdot A = B$ iff $\#A = \#B$ (we'll call $\binom{X}{k} = \{A \subset X \mid \#A = k\}$ an *orbit* of the action of S_X on $P(X)$).
   SUPP When X is infinite, $\binom{X}{\kappa}$ are orbits if $\kappa < |X|$, but there are multiple orbits on $\binom{X}{|X|}$, parametrized by the cardinality of the complement.
   (b) Let $A \subset X$. Show that $\mathrm{Stab}_{S_X}(A) \stackrel{\mathrm{def}}{=} \{\sigma \in S_X \mid \sigma \cdot A = A\} \simeq S_A \times S_{X-A}$ (we call $\mathrm{Stab}_{S_X}(A)$ the "stabilizer" of A).
   (c) Compute the index $[S_X : \mathrm{Stab}_{S_X}(A)]$. Now read Proposition 176 in the notes and use it to show that $\#\binom{X}{k} = \frac{n!}{k!(n-k)!}$.

Supplementary problem: Conjugation

A. Let G be a finite group.
   (a) Suppose all elements of G are conjugate. Show that $G = \{e\}$.
   (b) Suppose G has exactly two conjugacy classes. Show that $G \simeq C_2$.
   (**c) Suppose G has exactly three conjugacy classes. Show that $G =\simeq C_3$ or $G \simeq S_3$.
   RMK There exists an infinite group in which all non-identity elements are conjugate.

**B. Show that for each k there is $N = N(k)$ such that every finite group with k conjugacy classes has order at most N.

(hint for 2(a)t: let $H = \{1, h\}$, let $g \in G$, and consider the element ghg^{-1})
(hint for 2(b): consider the index of N)
(hint for 2(c): restrict $\mathrm{sgn}\colon S_n \to C_2$ to N)
(hint for 6(b): the number of conjugates of an element divides the order of the group)