Lior Silberman's Math 322: Problem Set 4 (due 5/10/2017)

Practice Problems

P1 Let G be a group with $|G| = 2$. Show that $G = \{e, g\}$ with $g \cdot g = e$ (hint: consider the multiplication table). Show that $G \simeq C_2$ (that is, find an isomorphism $C_2 \to G$).

P2 Let G be a group. Give a bijection between $\{H < G \mid \#H = 2\}$ and $\left\{g \in G \mid g^2 = e, g \neq e\right\}$.

P3. Are these groups? In each case either prove the group axioms or show that an axiom fails.
(a) The non-negative real numbers with the operation $x * y = \max\{x, y\}$.
(b) $\mathbb{R} \setminus \{-1\}$ with the operation $x * y = x + y + xy$.

P4 (Basics of groups and homomorphisms) Fix groups G, H, K and let $f \in \mathrm{Hom}(G, H)$, .
(a) Given also $g \in \mathrm{Hom}(H, K)$, show that $g \circ f \in \mathrm{Hom}(G, K)$.
(b) Suppose f is bijective. Then $f^{-1} \colon H \to G$ is a homomorphism.

Groups and Homomorphisms

1. Let G be a group, and let $(A, +)$ be an abelian group. For $f, g \in \mathrm{Hom}(G, A)$ and $x \in G$ define $(f+g)(x) = f(x) + g(x)$ (on the right this is addition in A).
(a) Show that $f + g \in \mathrm{Hom}(G, A)$.
(b) Show that $(\mathrm{Hom}(G, A), +)$ is an abelian group.
(*c) Let G be a group, and let $\mathrm{id} \colon G \to G$ be the identity homomorphism. Define $f \colon G \to G$ by $f(x) = (\mathrm{id}(x))(\mathrm{id}(x)) = x \cdot x = x^2$. Suppose that $f \in \mathrm{Hom}(G, G)$. Show that G is commutative.

2. (External Direct products) Let G, H be groups.
(a) On the product set $G \times H$ define an operation by $(g, h) \cdot (g', h') = (gg', hh')$. Show that $(G \times H, \cdot)$ is a group.
DEF this is called the (external) *direct product* of G, H.
(b) Let $\tilde{G} = \{(g, e_H) \mid g \in G\}$ and $\tilde{H} = \{(e_G, h) \mid h \in H\}$. Show that $\tilde{G}, \tilde{H}$ are subgroups of $G \times H$ and that $\tilde{G} \cap \tilde{H} = \{e_{G \times H}\}$.
SUPP Show that $\tilde{G}$, $\tilde{H}$ are isomorphic to G, H respectively.
(c) Show that for any $x = (g, h) \in G \times H$ we have $x\tilde{G}x^{-1} = \tilde{G}$ and $x\tilde{H}x^{-1} = \tilde{H}$ (the notation means $x\tilde{G}x^{-1} = \left\{xgx^{-1} \mid g \in \tilde{G}\right\}$).
EXAMPLE The Chinese remainder theorem shows that $C_n \times C_m \simeq C_{nm}$ if $\gcd(n, m) = 1$.

3. Products with more than two factors can be defined recursively, or as sets of k-tuples.
SUPP Find "natural" isomorphisms $G \times H \simeq H \times G$ and $(G \times H) \times K \simeq G \times (H \times K)$. We therefore write products without regard to the order of the factors.
DEF Write G^k for the k-fold product of groups isomorphic to G.
(a) Show that every non-identity element of C_2^k has order 2.
(b) Show that $C_3 \times C_3 \not\simeq C_9$.

4. The *Klein group* or the *four-group* is the group $V \simeq C_2 \times C_2$.
PRAC Check that $(\mathbb{Z}/12\mathbb{Z})^\times \simeq V$ and that $(\mathbb{Z}/8\mathbb{Z})^\times \simeq V$.
(a) Write a multiplication table for V, and show that V is not isomorphic to C_4.
(b) Show that $V = H_1 \cup H_2 \cup H_3$ where $H_i \subset V$ are subgroups isomorphic to C_2.

(c) Let G be a group of order 4. Show that G is isomorphic to either C_4 or to $C_2 \times C_2$.

5. Let G be a group, and let $H, K < G$ be subgroups and suppose that $H \cup K$ is a subgroup as well. Show that $H \subset K$ or $K \subset H$.

Extra credit

6. Show that, for each $d|n$, $\mathbb{Z}/n\mathbb{Z}$ has a unique subgroup of order (=size) d (and that the subgroup is cyclic).

7**. Let G be a finite group of order n, and suppose that for each $d|n$ G has at most one subgroup of order d. Show that G is cyclic.

Supplementary Problems

A. Let G be the *isometry group* of the Euclidean plane: $G = \{f\colon \mathbb{R}^n \to \mathbb{R}^n \mid \|f(\underline{x}) - f(\underline{y})\| = \|\underline{x} - \underline{y}\|\}$.
   (a) Show that every $f \in G$ is a bijection and that G is closed under composition and inverse.
   (b) For $\underline{a} \in \mathbb{R}^n$ set $t_{\underline{a}}(\underline{x}) = \underline{x} + \underline{a}$. Show that $t_{\underline{a}} \in G$, and that $\underline{a} \mapsto t_{\underline{a}}$ is an injective group homomorphism $(\mathbb{R}^n, +) \to G$.
   DEF Call the image the subgroup of *translations* and denote it by T.
   (c) Let $K = \{g \in G \mid g(\underline{0}) = \underline{0}\}$. Show that $K < G$ is a subgroup (we usually denote it $\mathrm{O}(n)$ and call it the *orthogonal group*).
   DEF This is called the *orthogonal group* and consists of rotations and reflections.
   FACT K acts on $\mathbb{R}^n$ by linear maps.
   (d) Show $\forall g \in G \exists t \in T : g\underline{0} = t\underline{0}$, and hence that $t^{-1}g \in K$. Conclude that $G = TK$.
   (e) Show that every $g \in G$ has a *unique* representation in the form $g = tk$, $t \in T$, $k \in K$ (hint: what is $T \cap K$?)
   (f) Show that K *normalizes* T: if $k \in K, t \in T$ we have $ktk^{-1} \in T$ (hint: use the linearity of k).
   (g) Show that $T \lhd G$: for every $g \in G$ we have $gTg^{-1} = T$.
   RMK We have shows that G is the *semidirect product* $G = K \ltimes T$.

B. Let X be a set of size at least 2, and fix $e \in X$. Define $*\colon X \times X \to X$ by $x * y = y$.
   (a) Show that $*$ is an associative operation and that e is a left identity.
   (b) Show that every $x \in X$ has a right inverse: an element $\bar{x}$ such that $x * \bar{x} = e$.
   (c) Show that $(X, *)$ is not a group.

C. Let $\{G_i\}_{i \in I}$ be groups. On the cartesian product $\prod_i G_i$ define an operation by

$$\left(\underline{g} \cdot \underline{h}\right)_i = g_i h_i$$

   (that is, by co-ordinatewise multiplication).
   (a) Show that $(\prod_i G_i, \cdot)$ is a group.
   DEF This is called the (external) *direct product* of the G_i.
   (b) Let $\pi_j\colon \prod_i G_i \to G_j$ be projection on the jth coordinate. Show that $\pi_j \in \mathrm{Hom}\left(\prod_i G_i, G_j\right)$.
   (c) (Universal property) Let H be any group, and suppose given for each i a homomorphism $f_i \in \mathrm{Hom}(H, G_i)$. Show that there is a unique homomorphism $\underline{f}\colon H \to \prod_i G_i$ such that for all i, $\pi_i \circ \underline{f} = f_i$.
   (**d) An *abstract direct product* of the groups G_i is a pair $\left(\mathbf{G}, \{q_i\}_{i \in I}\right)$ where $\mathbf{G}$ is a group, $q_i\colon \mathbf{G} \to G_i$ are homomorphisms, and the property of (c) holds. Suppose that $\mathbf{G}, \mathbf{G}'$ are both abstract direct products of the same family $\{G_i\}_{i \in I}$. Show that $\mathbf{G}, \mathbf{G}'$ are isomorphic (hint: the system $\{q_i\}$ and the universal property of $\mathbf{G}'$ give a map $\varphi\colon \mathbf{G} \to \mathbf{G}'$, and the same idea gives a map $\psi\colon \mathbf{G}' \to \mathbf{G}$. To see that the composition is the identity compare for example $q_i \circ \psi \circ \varphi$, $q_i \circ \mathrm{id}_{\mathbf{G}}$ and use the uniqueness of (c).

D. Let V, W be two vector spaces over a field F. On the set of pairs $V \times W = \{(\underline{v}, \underline{w}) \mid \underline{v} \in V, \underline{w} \in W\}$ define $(\underline{v}_1, \underline{w}_1) + (\underline{v}_2, \underline{w}_2) = (\underline{v}_1 +_V \underline{v}_2, \underline{w}_1 +_W \underline{w}_2)$ and $a \cdot (\underline{v}_1, \underline{w}_1) = (a \cdot_V \underline{v}_1, a \cdot_W \underline{w}_1)$.

(a) Show that this endows $V \times W$ with the structure of a vector space. This is called the *external direct sum* of V, W and denote it $V \oplus W$.

(b) Generalize the construction to an infinite family of vector spaces as in problem C(a).

(*c) State a universal property analogous to that of C(c),C(d) and prove the analogous results.

E. (Supplement to P3) Let $S^1 \subset \mathbb{R}^2$ be the unit circle. Then $f\colon [0, 2\pi) \to S^1$ given by $f(\theta) = (\cos\theta, \sin\theta)$ is continuous, 1-1 and onto but its inverse is not continuous.