Math 538: Commutative Algebra Problem Set

This problem set is for those who want to dig deeper. We may use some of those results in class, or only in problem sets.

Zorn's Lemma

DEFINITION. Let $\mathcal{F}$ be a set of sets. A *chain* in F is a subset $\mathcal{C} \subset \mathcal{F}$ such that for any $A, B \in \mathcal{C}$ either $A \subset B$ or $B \subset A$. An element $M \in \mathcal{F}$ is *maximal* if it is not contained in any other member.

AXIOM (Zorn's Lemma). *Let $\mathcal{F}$ be non-empty. Suppose that for any chain $\mathcal{C} \subset \mathcal{F}$ the set $\bigcup \mathcal{C} = \bigcup_{A \in \mathcal{C}} A$ also belongs to $\mathcal{F}$. Then $\mathcal{F}$ has maximal elements.*

1. Let F be a field, V a vector space over F. Let $\mathcal{F}$ be the family of linearly independent subsets of V. Show that $\mathcal{F}$ has maximal elements and conclude that V has a basis.

2. Let R be a ring (recall that rings here are commutative with identity), $I \subset R$ a proper ideal. Show that there exists a maximal ideal M of R containing I.

3. Let R be a ring, $S \subset R \setminus \{0\}$ a subset closed under multiplication. Show that there is a prime ideal P disjoint from S.

OPT Let $(X, \leq)$ be a partially ordered set (that is, $\leq$ is transitive and reflexive, and $x \leq y \wedge y \leq x \to x = y$). A *chain* in X is a subset $Y \subset X$ such that any two elements of Y are comparable (if $x, y \in Y$ then at least one of $x \leq y, y \leq x$ holds). An *upper bound* for a chain Y is an element $x \in X$ satisfying $y \leq x$ for all $y \in Y$. Show: suppose every chain in X has an upper bound. Then X has maximal elements.

Primes and Localization

Fix a commutative ring R. A *multiplicative subset* of R is a subset $S \subset R \setminus \{0\}$ closed under multiplication such that $1 \in S$. Fix such a subset.

4. Consider the following relation on $R \times S$: $(r, s) \sim (r', s') \iff \exists t \in S : t(s'r - sr') = 0$ (the intended interpretation of the pair (r, s) is as the fraction $\frac{r}{s}$).
   (a) Show that this is an equivalence relation, and that $(1, 1) \not\sim (0, 1)$.
   DEF Let $[r, s]$ (or $\frac{r}{s}$) denote the equivalence class of (r, s), and let $R[S^{-1}]$ denote the set of equivalence classes. Let $\iota \colon R \to R[S^{-1}]$ denote the map $\iota(r) = [r, 1]$.
   (b) Define $[r, s] + [r', s'] = [rs' + r's, ss']$ and $[r, s] \cdot [r', s'] = [rr', ss']$. Show that this defines a ring structure on $R[S^{-1}]$ and that ι is a ring homomorphism such that $\iota(S) \subset R[S^{-1}]^{\times}$. Show that ι is injective iff S contains no zero divisors.
   (c) Show that for any ring T and any homomorphism $\varphi \colon R \to T$ such that $\varphi(S) \subset T^{\times}$ there is a unique $\varphi' \colon R[S^{-1}] \to T$ such that $\varphi = \varphi' \circ \iota$.
   (d) Let $I \lhd R[S^{-1}]$ be a proper ideal. Show that $\iota^{-1}(I)$ is a proper ideal of R disjoint from S, and that I is the ideal of $R[S^{-1}]$ generated by $\iota(\iota^{-1}(I))$.
   (e) Conclude that when $S = R \setminus P$ for a prime ideal P (why is this closed under multiplication?) the ring $R[S^{-1}]$ is *local*: it has a unique maximal ideal (that being the ideal generated by the image of P).

DEFINITION. We call $R[S^{-1}]$ the *localization of R away from S*. If $S = R \setminus P$ for a prime ideal P we write R_P for $R[S^{-1}]$ and call it the localization of R *at P*.

5. Now let M be an R-module. On $M \times S$ define the relation $(m,s) \sim (m',s') \iff \exists t \in S : t(s'm - sm') = 0$ (with the interpretation $\frac{1}{s}m$).
   (a) Show that this is an equivalence relation, and that setting $[m,s] + [m',s'] = [s'm + sm', ss']$ and $[r,s] \cdot [m,s'] = [rm, ss']$ gives $M[S^{-1}]$, the set of equivalence classes, the structure of an $R[S^{-1}]$-module.
   (b) Let $\varphi\colon M \to N$ be a map of R-modules. Show that mapping $[m,s] \to [\varphi(m),s]$ gives a well-defined map $\varphi_{S^{-1}} : M[S^{-1}] \to N[S^{-1}]$ of $R[S^{-1}]$-modules.
   (c) Show that $\varphi_{S^{-1}}$ is surjective if φ is.
   (d) Show that $\operatorname{Ker}\varphi_{S^{-1}} = \left\{[m,s] \in M[S^{-1}] \mid \exists t \in S : tm \in \operatorname{Ker}\varphi\right\}$.

6. (The key proposition)
   (a) Let M be a non-zero R-module. Show that there is a prime P (in fact, a maximal ideal) such that M_P is a non-zero R_P-module.
   (b) Let $M \subset N$ be R modules. Show that $M \neq N$ iff there is a prime P such that $M_P \neq N_P$.

7. (Examples)
   (a) Let R be an integral domain. Show that $K(R) = R_{(0)}$ is a field. This is known as the *fraction field* of R. Show that in this case $R[S^{-1}]$ is isomorphic to the subring of $K(R)$ genreated by the image of R and of the inverses of the elements of S.
   (b) Let p be a rational prime. Show that the $\mathbb{Z}_{(p)}$ is a *discrete valuation ring*: that for every $x \in \mathbb{Q}^\times$ at least one of x, x^{-1} belongs to $\mathbb{Z}_{(p)}$.
   (c) Let $\Lambda < \mathbb{Z}^d$ be a subgroup of finite index, and let $\iota\colon \Lambda \to \mathbb{Z}^d$ be the incusion map. Show that $\iota_{(p)} : \Lambda_{(p)} \to \left(\mathbb{Z}_{(p)}\right)^d$ is an isomorphism iff p does not divide the index.

Integrality in general: A tour in commutative algebra

DEFINITION. Let $A \subset B$ be an extension of rings. $\beta \in B$ is said to be *integral* over A if $p(\beta) = 0$ for some monic $p \in A[x]$.

8. (Basic properties)
   (a) $\beta \in B$ is integral over B iff $A[\beta]$ is a finitely generated A-module iff there is a finitely generated A-module $M \subset B$ such that $\alpha M \subset M$.
   (b) Let $\alpha, \beta \in B$ be integral over A. Then so is every element of $A[\alpha,\beta]$
   (c) The set of elements in B integral over A is a subring of B called the *integral closure* of A in B, and denoted $\bar{A}$. Say that A is *integrally closed in B* if $\bar{A} = A$ (say an integral domain is *integrally closed* if it is integrally closed in its field of fractions).

9. Let $A \subset B \subset C$ be a rings.
   (a) Suppose B is integral over A and $\gamma \in C$ is integral over B. Then γ is integral over A.
   COR Let $\gamma \in C$ be integral over the integral closure of A in B. Then it is integral over A.
   COR Suppose A is integrally closed in B and B is integrally closed in C. Then A is integrally closed in C.
   (b) Let L/K be an extension of number fields. Then $\mathcal{O}_L$ is the integral closure of $\mathcal{O}_K$ in L.

Valuation rings

DEFINITION. An integral domain R is a *valuation ring* if for every $x \in K(R)$ at least one of x, x^{-1} belongs to R.

Fix a valuation ring R with field of fractions K.

10. (Basic properties)
    (a) Suppose that $a, b \in R$ and that $a+b$ is invertible. Show that one of a, b is invertible.
    (b) Conclude that the difference $R \setminus R^{\times}$ is an ideal of R, and hence that R has a unique maximal ideal.
    (c) Show that R is integrally closed.
    (d) Show that the set of ideals of R is a chain under inclusion.

DEFINITION. Say that the valuation ring R is *discrete* (a *dvr*) if all its non-zero ideals are powers of the maximal ideals.

11. Let R be a dvr with maximal ideal $\mathfrak{p}$.
    (a) Let $\varpi \in \mathfrak{p} \setminus \mathfrak{p}^2$. Show that $\mathfrak{p} = \varpi R$ and conclude that R is a PID.

Hints

For 6a: Let $m \in M$ be non-zero. Check that $\mathrm{Ann}(m) = \{r \in R \mid rm = 0\}$ is a proper ideal and localize at a maximal ideal containing it.

For 6b: Localize at P so that $(N/M)_P \neq 0$.

For 10a: Suppose first that $\frac{a}{b} \in R$, and use that $\frac{1}{a+b} \in R$ to invert one of a, b.

For 10c: Suppose that $\sum_{i=0}^{d-1} a_i x^i + x^d = 0$ for $a_i \in R$, $x \in K$. Show that $x \in R[x^{-1}]$ and conclude that $x \in R$.

For 10d: Let I, J be ideals and let $i \in I \setminus J$, $j \in J \setminus I$. Then $\frac{i}{j}, \frac{j}{i} \notin R$.