Lior Silberman's Math 412: Categories

In the second half of the 20th century it became clear that, in some sense, it is the *functions* that are important in the theory of an algebraic structure more than the structures themselves. This has been formalized in *Category Theory*, and the categorical point of view has been underlying much of the constructions in 412. Here's a taste of the ideas.

First examples

1. Some ideas of linear algebra can be expressed purely in terms of linear maps.
   (a) Show that $\{\underline{0}\}$ is the unique (up to isomorphism) vector space U such that for all vector spaces Z, $\mathrm{Hom}_F(U,Z)$ is a singleton.
   (b) Show that $\{\underline{0}\}$ is the unique (up to isomorphism) vector space U such that for all vector spaces Z, $\mathrm{Hom}_F(Z,U)$ is a singleton.
   (c) Let $f \in \mathrm{Hom}_F(U,V)$. Show that f is injective iff for all vector spaces Z, and all $g_1, g_2 \in \mathrm{Hom}(Z,U)$, $f \circ g_1 = f \circ g_2$ iff $g_1 = g_2$.
   (d) Let $f \in \mathrm{Hom}_F(U,V)$. Show that f is surjective iff for all vector spaces Z, and all $g_1, g_2 \in \mathrm{Hom}(V,Z)$, $g_1 \circ f = g_2 \circ f$ iff $g_1 = g_2$.
   (e) Show that $U \oplus V$ (the vector standard space structure on $U \times V$, together with the map $\underline{u} \mapsto (\underline{u}, \underline{0})$and $\underline{v} \mapsto (\underline{0}, \underline{v})$) has the property that for any vector space Z, the map $\mathrm{Hom}_F(U \oplus V, Z) \to \mathrm{Hom}_F(U,Z) \times \mathrm{Hom}_F(V,Z)$ given by restriction is a linear isomorphism.
   (f) Suppose that the triple (W, ι_U, ι_V) of a vector space W and maps from U,V to W respectively satisfies the property of (e). Show that there is a unique isomorphism $\varphi\colon W \to U \oplus V$ such that $\varphi \circ \iota_U$ is the inclusion of U in $U \oplus V$, and similarly for V.

2. (The category of sets)
   (a) Show that $\emptyset$ is the unique set U such that for all sets X, X^U is a singleton.
   (b) Show that $1 = \{\emptyset\}$ is the (up to bijection) set U such that for all sets X, U^X is a singleton.
   (c) Let $f \in Y^X$. Show that f is 1-1 iff for all sets Z, and all $g_1, g_2 \in X^Z$, $f \circ g_1 = f \circ g_2$ iff $g_1 = g_2$.
   (d) Let $f \in Y^X$. Show that f is onto iff for all sets Z, and all $g_1, g_2 \in Z^Y$, $g_1 \circ f = g_2 \circ f$ iff $g_1 = g_2$.
   (e) Given sets X_1, X_2 show that the disjoint union $X_1 \sqcup X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$ together with the maps $\iota_j(x) = (x, j)$ has the property that for any Z, the map $Z^{X_1 \sqcup X_2} \to Z^{X_1} \times Z^{X_2}$ given by restriction: $f \mapsto (f \circ \iota_1, f \circ \iota_2)$ is a bijection. If X_1, X_2 are disjoint, show that $X_1 \cup X_2$ with ι_j the identity maps has the same property.
   (f) Suppose that the triple (U, ι'_1, ι'_2) of a set U and maps $\iota'_j\colon X_j \to U$ satisfies the property of (e). Show that there is a unique bijection $\varphi\colon U \to X_1 \sqcup X_2$ such that $\varphi \circ \iota'_j = \iota_j$.

Categories

Roughly speaking, the "category of Xs" consists of all objects of type X, for each two such objects of all relevant maps between them, and of the *composition rule* telling us who to compose maps between Xs. We formalize this as follows:

DEFINITION. A *category* is a triple $\mathcal{C} = (V, E, h, t, \circ, \mathrm{Id})$ where: V is a class called the *objects* of $\mathcal{C}$, E is a class called the *arrows* of $\mathcal{C}$, $h, t \colon E \to V$ are maps assigning to each arrow its "head" and "tail" objects, $\mathrm{Id} \colon V \to E$ is map, and $\circ \subset (E \times E) \times E$ is a partially defined function (see below) called *composition*. We suppose that for each two objects $X, Y \in V$, $\mathrm{Hom}_{\mathcal{C}}(X,Y) \overset{\mathrm{def}}{=} \{e \in E \mid t(e) = X, h(e) = Y\}$ is a set, and then have:

- For $f, g \in E$, $f \circ g$ is defined iff $h(g) = t(e)$, in which case $f \circ g \in \mathrm{Hom}_{\mathcal{C}}(t(g), h(f))$.
- For each $X \in V$, $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ and for all f, $f \circ \mathrm{Id}_{t(f)} = \mathrm{Id}_{h(f)} \circ f = f$.
- $\circ$ is associative, in the sense that one of $(f \circ g) \circ h$ and $f \circ (g \circ h)$ is defined then so is the other, and they are equal.

In other words, for each three objects X, Y, Z we have a map $\circ \colon \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ which is associative, and respects the distinguished "identity" map.

EXAMPLE. Some familiar categories:

- **Set**: the category of sets. Here $\mathrm{Hom}_{\mathbf{Set}}(X,Y) = Y^X$ is the set of set-theoretic maps from X to Y, composition is composition of functions, and Id_X is the identity map $X \to X$.
- **Top**: the category of topological spaces with continuous maps. Here $\mathrm{Hom}_{\mathbf{Top}}(X,Y) = C(X,Y)$ is the set of continuous maps $X \to Y$.
- **Grp**: the category of groups with group homomorphims. $\mathrm{Hom}_{\mathbf{Grp}}(G,H)$ is the set of group homomorphisms.
- **Ab**: the category of abelian groups. Note that for abelian groups A, B we have $\mathrm{Hom}_{\mathbf{Ab}}(A,B) = \mathrm{Hom}_{\mathbf{Grp}}(A,B)$ [the word for this is "full subcategory"]
- $\mathbf{Vect}_F$: the category of vector spaces over the field F. Here $\mathrm{Hom}_{\mathbf{Vect}_F}(U,V) = \mathrm{Hom}_F(U,V)$ is the space of linear maps $U \to V$.

3. (Formalization of familiar words) For each category above (except **Set**) express the statement $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ $\circ \colon \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z)$ as a familiar lemma. For example, "the identity map $X \to X$ is continuous" and "the composition of continuous functions is continuous".

Properties of a single arrow and a single object

DEFINITION. Fix a category $\mathcal{C}$, objects $X,Y \in \mathcal{C}$, and an arrow $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

- Call f a *monomorphism* if for every object Z and every two arrows $g_1, g_2 \in \mathrm{Hom}(Z,X)$ we have $f \circ g_1 = f \circ g_2$ iff $g_1 = g_2$.
- Call f an *epimorphism* if for every object Z and every two arrows $g_1, g_2 \in \mathrm{Hom}(Y,Z)$ we have $g_1 \circ f = g_2 \circ f$ iff $g_1 = g_2$.
- Call f an *isomorphism* if there is an arrow $f^{-1} \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that $f^{-1} \circ f = \mathrm{Id}_X$ and $f \circ f^{-1} = \mathrm{Id}_Y$.

4. Show that two sets are isomorphic iff they have the same cardinality,

5. Suppose that f is an isomorphism.
   (a) Show that f^{-1} is an isomorphism.
   (b) Show that f is a monomorphism and an epimorphism.
   (c) Show that there is a unique $g \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ satisfying the properties of f^{-1}.
   (d) Show that composition with f gives bijections $\mathrm{Hom}_{\mathcal{C}}(X,Z) \to \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ and $\mathrm{Hom}_{\mathcal{C}}(W,X) \to \mathrm{Hom}_{\mathcal{C}}(W,Y)$ which respect composition.
   RMK Part (d) means that isomorphic objects "are the same" as far as the category is concerned.

6. For each category in the example above:
   (a) Show that f is a monomorphism iff it is injective set-theoretically.
   (b) Show that f is an epimorphism iff it is surjective set-theoretically, *except* in **Top**.
   (c) Which continuous functions are epimorphisms in **Top**?

DEFINITION. Call an object $I \in \mathcal{C}$ *initial* if for every object X, $\mathrm{Hom}_{\mathcal{C}}(I,X)$ is a singleton. Call $F \in \mathcal{C}$ *final* if for every $X \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(X,F)$ is a singleton.

7. (Uniqueness)
   (a) Let I_1, I_2 be initial. Show that there is a unique isomorphism $f \in \mathrm{Hom}_{\mathcal{C}}(I_1, I_2)$.
   (b) The same for final objects.

8. (Existence)
   (a) Show that the $\emptyset$ is initial and $\{\emptyset\}$ is final in **Set**. Why is $\{\emptyset\}$ not an initial object?
   (b) Show that $\{\underline{0}\}$ is both initial and final in $\mathbf{Vect}_F$.
   (c) Find the initial and final objects in the categories of groups and abelian groups.

Sums and products

DEFINITION. Let $\{X_i\}_{i\in I} \subset \mathcal{C}$ be objects.

- Their *coproduct* is an object $U \in \mathcal{C}$ together with maps $u_i \in \mathrm{Hom}_{\mathcal{C}}(X_i, U)$ such that for every object Z the map $\mathrm{Hom}(U,Z) \to \times_{i\in I}\mathrm{Hom}_{\mathcal{C}}(X_i,U)$ given by $f \mapsto (f\circ u_i)_{i\in I}$ is a bijection.
- Their *product* is an object $P \in \mathcal{C}$ together with maps $p_i \in \mathrm{Hom}_{\mathcal{C}}(P, X_i)$ such that for every object Z the map $\mathrm{Hom}(Z,U) \to \times_{i\in I}\mathrm{Hom}_{\mathcal{C}}(U,X_i)$ given by $f \mapsto (p_i \circ f)_{i\in I}$ is a bijection.

9. (Uniqueness)
   (a) Show that if U, U' are coproducts then there is a unique isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(U,U')$ such that $\varphi \circ u_i = u'_i$.
   (b) Show that if P, P' are products then there is a unique isomorphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(P,P')$ such that $p'_i \circ \varphi = p_i$.

10. (Existence)
    (a) In the category **Set**.
        (i) Show that the *disjoint union* $\bigsqcup_i X_i \overset{\text{def}}{=} \bigcup_{i\in I}(X_i \times \{i\})$ with maps $u_i(x) = (x,i)$ is a coproduct. In particular, if X_i are disjoint show that $\bigcup_{i\in I} X_i$ is a coproduct.
        (ii) Show that $\times_{i\in I} X_i$ with maps $p_j((x_i)_{i\in I}) = x_j$ is a product.
    (b) In the category **Top**.
        (i) Show that $[0,2) = [0,1) \cup [1,2)$ (with the inclusion maps) is a coproduct in **Set** but not in **Top** (subspace topologies from $\mathbb{R}$).
        (ii) Show that $\bigsqcup_i X_i$ with the topology $\mathcal{T} = \{\bigcup_{i\in I}(A_i \times \{i\}) \mid A_i \subset X_i \text{ open}\}$ is a coproduct .
        (iii) Show that the product topology on $\times_{i\in I} X_i$ makes it into a product.
    (c) In the category $\mathbf{Vect}_F$.
        (i) Show that $\oplus_{i\in I} X_i$ is a coproduct .
        (ii) Show that $\prod_{i\in I} X_i$ is a product.
    (d) In the category **Grp**.
        (i) Show that the "coordinatewise" group structure on $\times_{i\in I} G_i$ is a product.
        – The coproduct exists, is called the *free product* of the groups G_i, and is denoted $*_{i\in I} G_i$.

Challenge

A category can be thought of as a "labelled graph" – it has a set of vertices (the objects), a set of directed edges (the arrows), and a composition operator and marked identity morphism, but in fact every vertex has a "label" – the object it represents, and every arrow similarly has a label. Suppose you are only given the combinatorial data, without the "labels" (imagine looking at the category of groups as a graph and then deleting the labels that say which vertex is which group). Can you restore the labels on the objects? Given that, can you restore the labels on the arrows? [up to automorphism of each object]?

This is easy in **Set**, not hard in **Top** and $\mathbf{Vect}_F$, a challenge in **Ab** and requires ingenuity in **Grp**.