Math 101 – SOLUTIONS TO WORKSHEET 30
POWER SERIES

(1) Which of the following is a power series:

$$\square \sum_{n=0}^{\infty} \frac{n!(x-3)^n}{2^{2^n}} \qquad \square \sum_{n=0}^{\infty} \frac{3}{n!} (e^x)^n$$

Solution: The first is a power series, the second isn't (there are powers of e^x, not powers of x!).

1. The interval of convergence

(2) Find the interval of convergence and radius of convergence of the power series

(a) $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{(x-1)^n}{n}$

Solution: We have $c=1$, $A_n=\frac{(-1)^n}{n}$. Now

$$\left|\frac{(-1)^n(x-1)^{n+1}}{n+1} \Big/ \frac{(-1)^{n-1}(x-1)^n}{n}\right| = |x-1|\left|\frac{n}{n+1}\right| = |x-1|\frac{1}{1+\frac{1}{n}} \xrightarrow[n\to\infty]{} |x-1|$$

so the series converges absolutely when $|x-1|<1$ and diverges when $|x-1|>1$. The series therefore converges at least on $(0,2)$.At the endpoint $x=2$ the series is $\sum_{n=1}^{\infty}(-1)^n\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ and it converges by the alternating series test. At $x=0$ we have the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ which is a divergent p-series ($p=1$). The interval of convergences is them $(0,2]$.

Solution: $L=\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{n}{n+1} = \lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = 1$ so $R=\frac{1}{L}=1$, and the series converges at least on $(c-R,c+R)=(0,2)$. At the endpoint $x=2$ the series is $\sum_{n=1}^{\infty}(-1)^n\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}$ and it converges by the alternating series test. At $x=0$ we have the series $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(-1)^n}{n} = -\sum_{n=1}^{\infty}\frac{1}{n}$ which is a divergent p-series ($p=1$). The interval of convergences is them $(0,2]$.

(b) $\sum_{n=0}^{\infty} n!x^n$

Solution: If $x\neq 0$ we have $\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = (n+1)|x| \xrightarrow[n\to\infty]{} \infty$ and the series diverges by the ratio test, so the series converges only for $x=0$.

Solution: We have $A_n=n!$ and $L=\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \lim_{n\to\infty}(n+1)=\infty$ so $R=\frac{1}{L}=0$ and the series only converges at $x=0$.

(c) $\sum_{n=0}^{\infty}\frac{x^n}{n!}$

Solution: We have $\left|\frac{x^{n+1}}{(n+1)!}\Big/\frac{x^n}{n!}\right| = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0$ so the series converges for all x. The interval is $(-\infty,\infty)$ and the radius is ∞.

Solution: We have $A_n=\frac{1}{n!}$ and $\lim_{n\to\infty}\left|\frac{A_{n+1}}{A_n}\right| = \lim_{n\to\infty}\frac{n!}{(n+1)!} = \lim_{n\to\infty}\frac{1}{n+1}=0$ so $R=\infty$ and the interval of convergence is $(-\infty,\infty)$.

(d) (Final, 2014, variant) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{3^n(n^2+1)}$

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Solution: We have

$$\begin{aligned}\left|\frac{(x-2)^{n+1}}{3^{n+1}\left((n+1)^2+1\right)}\Big/\frac{(x-2)^n}{3^n\left(n^2+1\right)}\right| &= \left|\frac{(x-2)^{n+1}}{(x-2)^n}\right|\cdot\left|\frac{3^n}{3^{n+1}}\right|\cdot\frac{n^2+1}{n^2+2n+2}\\ &= |x-2|\cdot\frac{1}{3}\cdot\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}\\ &\xrightarrow[n\to\infty]{} \frac{|x-2|}{3}.\end{aligned}$$

so the series converges for $\frac{|x-2|}{3}<1$ and diverges for $\frac{|x-2|}{3}>1$. We rewrite the first interval as $|x-2|<3$ so the radius of convergence is $R=3$. The endpoints of the interval of convergence are then $2\pm3=-1,5$. At $x=5$ we have the series $\sum_{n=0}^{\infty}\frac{1^n}{n^2+1}=\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ which convergeces by comparison to the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (we have $\frac{1}{n^2+1}<\frac{1}{n^2}$ for all $n\geq1$). At $x=-1$ we have the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ which converges absolutely by the convergence at $x=5$. The interval of convergence is thus $[-1,5]$.

Solution: We have $L=\lim_{n\to\infty}\left(\frac{1}{3^{n+1}((n+1)^2+1)}\Big/\frac{1}{3^n(n^2+1)}\right)=\lim_{n\to\infty}\frac{1}{3}\frac{n^2+1}{n^2+2n+2}=\lim_{n\to\infty}\frac{1}{3}\frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}}=\frac{1}{3}$ so the radius of convergence is $R=\frac{1}{1/3}=3$. The endpoints of the interval of convergence are then $2\pm3=-1,5$. At $x=5$ we have the series $\sum_{n=0}^{\infty}\frac{1^n}{n^2+1}=\sum_{n=0}^{\infty}\frac{1}{n^2+1}$ which convergeces by comparison to the p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ (we have $\frac{1}{n^2+1}<\frac{1}{n^2}$ for all $n\geq1$). At $x=-1$ we have the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+1}$ which converges absolutely by the convergence at $x=5$. The interval of convergence is thus $[-1,5]$.

(e) (Final, 2011) $\sum_{n=0}^{\infty}\frac{(x-2)^n}{\log(n+2)}$

Solution: We have

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{1}{\log(n+3)}\Big/\frac{1}{\log(n+2)}\right) &= \lim_{n\to\infty}\frac{\log(n+2)}{\log(n+3)}=\lim_{x\to\infty}\frac{\log(x+2)}{\log(x+3)}\\ &= \lim_{x\to\infty}\frac{1}{x+2}\Big/\frac{1}{x+3}=\lim_{x\to\infty}\frac{x+3}{x+2}\\ &= \lim_{x\to\infty}\frac{1+\frac{3}{x}}{1+\frac{2}{x}}=1\end{aligned}$$

so the radius of convergence is $R=\frac{1}{1}=1$. The endpoints of the interval of convergence are then $2\pm1=1,3$. At $x=3$ we have the series $\sum_{n=0}^{\infty}\frac{1^n}{\log(n+2)}=\sum_{n=0}^{\infty}\frac{1}{\log(n+2)}$ which diverges by comparison to the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ (we have $\log(n+2)<n$ for all large n, for example because $\lim_{x\to\infty}\frac{\log(x+2)}{x}=\lim_{x\to\infty}\frac{1}{x+2}\cdot\frac{1}{1}=0$, so $\frac{1}{\log(n+2)}>\frac{1}{n}$ eventually). At $x=1$ we have the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{\log(n+2)}$ which converges by alternating series test (the signs change, and $\log(n+2)$ increases monotonically to infinity so $\frac{1}{\log(n+2)}$ decreases monotonically to zero).

(3) Consider a power series $\sum_{n=0}^{\infty}A_n(x-5)^n$.

(a) The power series converges at $x=-3$. Show that it converges at $x=10$.

Solution: Since $|-3-5|=8$, the radius of convergence is at least 8. Since $|10-5|=5<8\leq R$, the series converges at 10. Note that the series may or may not converge at 13 (it may be that -5 and 13 are the two endpoints of the interval of convergence).

(b) The power series diverges at $x=15$. Show that it diverges at $x=-15$.

Solution: Since $|15-5|=10$, the radius of convergence is at most 10. Since $|-15-5|=20>10\geq R$, the series diverges at -15. Note that the series may or may not converge at 5 (it may be that 5 and 15 are the two endpoints of the interval of convergence).

(c) Can you tell if the series converges at $x=14$? What can you say about the radius of convergence?

Solution: We have learned that the radius of convergence satisfies $8\leq R\leq10$. Since $|14-5|=9$ it is impossible to tell whether 14 lies in the interval of convergence.

2. Manipulating power series

(4) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$.

(a) Find the power series representation of $f'(x)$. What is $f(x)$?

Solution: $f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)$ so $f'(x) = f(x)$ and $f(x) = Ce^x$. Since $f(0) = 1$, we have $C = 1$ and $f(x) = e^x$.

(b) Find the power series representation of $g'(x)$. What is $g'(x)$? What is $g(x)$?

Solution: $g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n x^{n-1}}{n} = \sum_{n=1}^{\infty} (-x)^{n-1} = \sum_{m=0}^{\infty} (-x)^m = \frac{1}{1-(-x)} = \frac{1}{1+x}$ so $g'(x) = \frac{1}{1+x}$ and $g(x) = \log(1+x) + C$. Since $g(0) = 0$, we have $C = 0$ and $g(x) = \log x$.

(c) Find the power series representation of $\int_0^x f(-t^2)\,\mathrm{d}t$.

Solution: We have $f(-t^2) = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}$. Integrating term-by-term we have

$$\int_0^x f(-t^2)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left[\frac{t^{2n+1}}{2n+1}\right]_{t=0}^{t=x} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}\,.$$