Math 101 – SOLUTIONS TO WORKSHEET 27
ALTERNATING SERIES

1. Converge or Diverge?

(1) Determine, with explanation, whether the following series converge or diverge.

(a) (Alternating harmonic series) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.
Solution: The terms are alternating in sign, decreasing in magnitude, and tending to zero, so by the alternating series test the series converges.

(b) $1 - \frac{1}{4} + \frac{1}{3} - \frac{1}{16} + \frac{1}{5} - \frac{1}{36} + \frac{1}{7} - \frac{1}{64} + \frac{1}{9} - \frac{1}{100} + \frac{1}{11} - \frac{1}{144} + \cdots$
Solution: The positive terms are $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$ and $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ is a divergent series since $\frac{1}{2n-1} \geq \frac{1}{2n} > 0$ and $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n}$ is the divergent harmonic series). The negative terms are $-\frac{1}{4} - \frac{1}{16} - \cdots = -\sum_{n=1}^{\infty} \frac{1}{(2n)^2} = -\frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2}$ and this is a convergent p-series ($p = 2 > 1$). Since the sum of a convergent series and a divergent series diverges, the series as a whole diverges.
NOTE: the series is alternating and the terms tend to zero but they are not decreasing in magnitude.

(c) (Final 2014) $\sum_{n=1}^{\infty} \frac{n\cos(\pi n)}{2^n}$
Solution: Since $\cos(\pi n) = (-1)^n$ the series is alternating. Let $f(x) = \frac{x}{2^x}$. Then $f'(x) > 0$ for $x > 0$, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{(\log 2)2^x} = 0$ by l'Hôpital and

$$f'(x) = \frac{2^x - x\log 2 \cdot 2^x}{(2^x)^2} = -\frac{(\log 2)x - 1}{2^x} < 0$$

for $x > \frac{1}{\log 2}$. It follows that $f'(x)$ is positive, eventually decreasing, and tends to zero. By the alternating series test, $\sum_{n=1}^{\infty}(-1)f(n)$ converges.

(d) (Final 2011) $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p} = 1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots$ (your answer will depend on p)
Solution: For $p > 0$, the numbers $\frac{1}{n^p}$ are decreasing as n increases and tend to zero, so the series converges by the alternating series test. For $p \leq 0$, the terms n^{-p} are all at least one, so the series diverges by the divergence test (the terms fail to converge to zero).

(2) Power series

(a) (Final 2013, variant) Decide whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}(x+2)^n$ converges or diverges at $x = -1$ and at $x = -3$.
Solution: At $x = -1$ we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$. The terms are alternating in sign, decreasing in magnitude, and tending to zero, so by the alternating series test the series converges at $x = -1$. At $x = -3$ we have the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}(-1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n}}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ which is a divergent p-series ($p = \frac{1}{2} \leq 1$).

(b) Decide whether the series $\sum_{n=1}^{\infty} nx^n$ converges or diverges at $x = 1$ and $x = -1$.
Solution: At both values the series diverges, since the terms tend to infinity in magnitude.

2. Error estimates

(3) (a) It is known that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \log 2$. How many terms are needed for the error to be less than 0.01?

Date: 15/3/2017, Worksheet by Lior Silberman.

Solution: The series is alternating, so the error in approximating its sum by a partial sum is less than the first ommitted term. Taking the first 99 terms, this means that

$$\left|\log 2 - \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{99}\right)\right| \leq \frac{1}{100}$$

as desired.

(b) It is known that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \cdots = \frac{\pi}{4}$. How many terms are needed for the error to be less than 0.001?

Solution: Again the series is alternating. The magnitude of the nth term is $\frac{1}{2n-1}$ so taking the first 500 terms we get that

$$\left|\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots - \frac{1}{999}\right)\right| \leq \frac{1}{1001} < \frac{1}{1000}\,.$$

(4) (MacLaurin expansions)

(a) It is known that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. How close is $\frac{1}{2} - \frac{1}{6} + \frac{1}{24}$ to $\frac{1}{e}$? How many terms are needed to approximate $\frac{1}{e}$ to within $\frac{1}{1000}$?

Solution: The series $e^{-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ is alternating, and $n!$ is increasing to infinity so that $\frac{1}{n!}$ monotonically decrease to zero. By the alternating series test, the error is bounded by the next term.

(a) The next term after $\frac{1}{24} = \frac{1}{4!}$ is $-\frac{1}{5!} = \frac{1}{120}$ so

$$\left|\frac{1}{e} - \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)\right| \leq \frac{1}{120}\,.$$

(b) If we want to approximate $\frac{1}{e}$ to within $\frac{1}{1000}$ we need to keep terms until one is smaller than than. We have $\frac{1}{6!} = \frac{1}{720}$ and $-\frac{1}{7!} = -\frac{1}{5040}$ so keeping the first seven terms we have

$$\left|\frac{1}{e} - \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720}\right)\right| \leq \frac{1}{5040} < \frac{1}{1000}\,.$$

(b) The *error function* is (roughly) given by $\mathrm{erf}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1}$. How many terms are needed to approximate $\mathrm{erf}(\frac{1}{10})$ to within 10^{-11}?

Solution: Using $x = \frac{1}{10}$ gives the series

$$\mathrm{erf}\left(\frac{1}{10}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)10^{2n+1}}\,.$$

Since each of the factors of $n!(2n+1)10^{2n+1}$ is increasing, the terms of the series terms are monotonically decreasing in magnitude, tending to zero, and are clearly alternating in sign. For $n = 4$ we have $n!(2n+1)10^{2n+1} = 24 \cdot 9 \cdot 10^9 > 100 \cdot 10^9 = 10^{11}$ since $24 \cdot 9 > 20 \cdot 5 = 100$. By the alternating series test taking the first four terms is sufficient:

$$\left|\mathrm{erf}\left(\frac{1}{10}\right) - \left(1 - \frac{1}{300} + \frac{1}{10^4} - \frac{1}{42 \cdot 10^7}\right)\right| < 10^{-11}\,.$$