Math 101 – SOLUTIONS TO WORKSHEET 26
THE COMPARISON TEST

1. Comparison by Massaging

(1) Determine, with explanation, whether the following series converge or diverge.

(a) (Final 2014) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

Solution: For $n \geq 1$ we have $n^2 + 1 \leq n^2 + n^2 = 2n^2$ so that $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{2n^2}} = \frac{1}{\sqrt{2}}\frac{1}{n}$. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test with $p = 1 \leq 1$) so by the comparison test the given series diverges as well.

Solution: (Really complicated) The function $f(x) = \frac{1}{\sqrt{x^2+1}}$ is positive and decreasing on $[0, \infty)$. Using the susbtitutions $x = \tan\theta$ and $\sin\theta = u$ we have:

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sqrt{1+\tan^2\theta}} \\
&= \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sec\theta} \\
&= \int \frac{\cos\theta\,\mathrm{d}\theta}{\cos^2\theta} \\
&= \int \frac{\mathrm{d}u}{1-u^2} = \frac{1}{2}\int \left[\frac{1}{1+u} + \frac{1}{1-u}\right] \mathrm{d}u \\
&= \frac{1}{2}\log|1+u| - \frac{1}{2}\log|1-u| + C \\
&= \frac{1}{2}\log\left|\frac{1+u}{1-u}\right| + C \\
&= \frac{1}{2}\log\frac{1+\sin\theta}{1-\sin\theta} + C
\end{aligned}$$

(we don't need absolute values since $1 + \sin\theta$ and $1 - \sin\theta$ are both non-negative). Now $\sin\theta = \frac{x}{\sqrt{1+x^2}}$ so

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &= \frac{1}{2}\log\frac{1+\frac{x}{\sqrt{1+x^2}}}{1-\frac{x}{\sqrt{1+x^2}}} + C \\
&= \frac{1}{2}\log\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} + C\,.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x} &= \frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x} \\
&= \frac{\left(\sqrt{1+x^2}+x\right)^2}{1+x^2-x^2} = \left(\sqrt{1+x^2}+x\right)^2
\end{aligned}$$

so finally

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &= \frac{1}{2}\log\left(\sqrt{1+x^2}+x\right)^2 + C \\
&= \log\left(\sqrt{1+x^2}+x\right) + C\,.
\end{aligned}$$

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We therefore have

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}x}{\sqrt{1+x^2}} &= \lim_{T\to\infty}\int_0^T \frac{\mathrm{d}x}{\sqrt{1+x^2}} \\ &= \lim_{T\to\infty}\left[\log\left(\sqrt{1+T^2}+T\right)-\log\left(\sqrt{1+0^2}+0\right)\right] \\ &= \lim_{T\to\infty}\log\left(\sqrt{1+T^2}+T\right)=\infty\,.\end{aligned}$$

By the integral test the series $\sum_{n=1}^\infty \frac{1}{\sqrt{1+n^2}}$ diverges as well.

(b) (Final 2013, variant) $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\frac{1}{121}+\cdots$
Solution: The series is $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$ and has positive terms. The nth odd number is $2n-1$ so the series is

$$\sum_{n=1}^\infty \frac{1}{(2n-1)^2}\,.$$

For $n\geq 1$, $2n-1\geq 2n-n=n$ so $\frac{1}{(2n-1)^2}\leq\frac{1}{n^2}$. The series $\sum_{n=1}^\infty\frac{1}{n^2}$ converges by the p-test ($p=2>1$) so by the comparison test our series converges too.

(c) (Final 2013) $\sum_{n=1}^\infty \frac{n+\sin n}{1+n^2}$
Solution: For $n\geq 2$ we have $n+\sin n\geq n-1\geq n-\frac{n}{2}$ and $1+n^2\leq 2n^2$ so that for $n\geq 2$ we have $\frac{n+\sin n}{n^2+1}\geq\frac{n/2}{2n^2}=\frac{1}{4n}$. The harmonic series $\sum_{n=1}^\infty\frac{1}{n}$ diverges (p-test with $p=1$) so by the comparison test the given series diverges as well.

(d) $1+\frac{1}{2^3}+\frac{1}{3^2}+\frac{1}{4^3}+\frac{1}{5^2}+\frac{1}{6^3}+\frac{1}{7^2}+\cdots$
Solution: Let a_n be the nth term of the series (which is positive) so that $a_n=\begin{cases}\frac{1}{n^2} & n\text{ odd}\\ \frac{1}{n^3} & n\text{ even}\end{cases}$.
For $n\geq 1$ we have $\frac{1}{n^3}\leq\frac{1}{n^2}$ so $a_n\leq\frac{1}{n^2}$ in any case. Now $\sum_{n=1}^\infty\frac{1}{n^2}$ converges by the p-test ($p=2>1$) so by the comparison test the series $\sum_{n=1}^\infty a_n$ converges as well.

2. Limit comparison test

(2) Determine, with explanation, whether the following series converge or diverge.

(a) (Final 2014) $\sum_{n=1}^\infty\frac{1}{\sqrt{n^2+1}}$
Solution: We have $\lim_{n\to\infty}\frac{1}{n}\Big/\frac{1}{\sqrt{n^2+1}}=\lim_{n\to\infty}\frac{\sqrt{n^2+1}}{n}=\lim_{n\to\infty}\sqrt{1+\frac{1}{n^2}}=1$. The harmonic series $\sum_{n=1}^\infty\frac{1}{n}$ diverges (p-test with $p=1$) so by the limit comparison test our series dinverges as well.

(b) (Final 2013, variant) $1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\frac{1}{121}+\cdots$
Solution: The series is $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots$ and has positive terms. The nth odd number is $2n-1$ so the series is

$$\sum_{n=1}^\infty \frac{1}{(2n-1)^2}\,.$$

Now $\lim_{n\to\infty}\frac{1}{n^2}\Big/\frac{1}{(2n-1)^2}=\lim_{n\to\infty}\left(\frac{2n-1}{n}\right)^2=\lim_{n\to\infty}\left(2-\frac{1}{n}\right)^2=4$. The series $\sum_{n=1}^\infty\frac{1}{n^2}$ converges by the p-test ($p=2>1$) so by the limit comparison test our series converges too.

(c) (Final 2013) $\sum_{n=1}^\infty\frac{n+\sin n}{1+n^2}$
Solution: We have

$$\lim_{n\to\infty}\frac{n+\sin n}{n^2+1}\Big/\frac{1}{n}=\lim_{n\to\infty}\frac{1+\frac{\sin n}{n^2}}{1+\frac{1}{n^2}}=\frac{1+\lim_{n\to\infty}\frac{\sin n}{n^2}}{1+\lim_{n\to\infty}\frac{1}{n^2}}$$

Now $\lim_{n\to\infty}\frac{1}{n^2}=0$. Since $-1\leq\sin n\leq 1$, we have $-\frac{1}{n^2}\leq\frac{\sin n}{n^2}\leq\frac{1}{n^2}$ and $\lim_{n\to\infty}\left(-\frac{1}{n^2}\right)=-\lim_{n\to\infty}\frac{1}{n^2}=0$ also so by the squeeze theorem, $\lim_{n\to\infty}\frac{\sin n}{n^2}=0$. It follows that

$$\lim_{n\to\infty}\frac{n+\sin n}{n^2+1}\Big/\frac{1}{n}=\frac{1+0}{1+0}=1\,.$$

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (p-test with $p = 1$) so by the limit comparison test the given series diverges as well.