Math 101 – WORKSHEET 25
THE INTEGRAL TEST

1. The integral test

(1) Decide if each series converges or diverges
   (a) $\sum_{n=1}^{\infty} \frac{n}{e^n}$

   (b) (Final 2014) $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ (your answer will depend on p!)

   (c) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$

(2) The integral $\int_2^{\infty} \frac{x+\sin x}{1+x^2}\,dx$ diverges. Why can't we use the integral test to conlcude that $\sum_{n=2}^{\infty} \frac{n+\sin n}{1+n^2}$ diverges as well?

Date: 10/3/2017, Worksheet by Lior Silberman.

2. Tail estimates (<u>**not examinable in Math 101**</u>)

Let $f(x)$ be positive and non-increasing on $[a, \infty)$. Then for $M > N > a$ we have

$$\sum_{n=N}^{M} f(n) \geq \int_N^{M+1} f(x)\,\mathrm{d}x \geq \sum_{n=N+1}^{M+1} f(n)$$

and hence

$$\sum_{n=N}^{\infty} f(n) \geq \int_N^{\infty} f(x)\,\mathrm{d}x \geq \sum_{n=N+1}^{\infty} f(n)$$

(3) Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$

(a) Show that $\sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \frac{1}{N}$.

(b) How many terms to we need to include to approximate the sum of the series within 10^{-5}?

(3) (The harmonic series)

(a) Show that $\sum_{n=1}^{N} \frac{1}{n} \geq \log(N+1)$

(b) Show that $\sum_{n=1}^{N} \frac{1}{n} \leq (1 - \log 2) + \log(N+1)$

(4) Bonus problem: $\gamma = \lim_{N\to\infty} \left(\sum_{n=1}^{N} \frac{1}{n} - \log(N+1)\right)$ exists.

(a) For $N \geq 1$ set $s_N = \sum_{n=1}^{N} \frac{1}{n} - \log(N+1)$ (set $s_0 = 0$) and let $a_n = s_n - s_{n-1}$. Show that $a_n = \frac{1}{n} - \log\left(1 + \frac{1}{n}\right)$.

(b) Show that there is $C > 0$ such that $0 \leq a_n \leq \frac{C}{n^2}$ for all $n \geq 1$. By the comparison test, $\sum_{n=1}^{\infty} a_n$ converges.

(c) Show that $s_N = \sum_{n=1}^{N} a_n$. It follows that $\{s_N\}_{n=1}^{\infty}$ converges.

The number γ is called the Euler–Mascheroni constant, its value is about 0.577.