Math 101 – SOLUTIONS TO WORKSHEET 21
SEPARABLE DIFFERENTIAL EQUATIONS

1. What is a DE?

(1) Consider the differential equation $y' = 3y^2$
  (a) For which values of C, D is $f(x) = Cx^D$ a solution to the equation?
  Solution: Plug in to get $CDx^{D-1} = 3C^2x^{2D}$ so we need $2D = D-1$ ($D = -1$) and $CD = 3C^2$ so either $C = 0$ ($f(x) = 0$ is a solution!) or $D = -1$ and $C = \frac{D}{3} = -\frac{1}{3}$.

$$\boxed{f(x) = -\frac{1}{3x}}$$

  is the solution.
  (b) Suppose $f(x)$ is a solution. Show that $f(x-a)$ is also a solution for any a. What is the solution with $f(0) = 1$?
  Solution: Let $f(x) = -\frac{1}{3(x-a)}$. Then $f'(x) = \frac{1}{3(x-a)^2}$ while $3\left(f(x)\right)^2 = \frac{3}{9(x-a)^2} = \frac{1}{3(x-a)^2}$ so indeed this is a solution. We need a such that $-\frac{1}{3(0-a)} = 1$, that is

$$\frac{1}{3a} = 1$$

  so $\boxed{a = \frac{1}{3}}$ and $f(x) = \frac{1}{1-3x}$ is the solution.

2. Separation of variables

(2) Solve the follwing equations using separation of variables
  (a) $y' = x^3$
  (b) $y' = 5y$
  (c) (Final, 2012) $y' = xy$, $y(0) = e$.
(3) (Final 2014) Find the solution of the DE $x\frac{\mathrm{d}y}{\mathrm{d}x} + y = y^2$ that satisfies $y(1) = -1$.
  Solution: Write this as $x\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - y$ so

$$\frac{\mathrm{d}y}{y^2 - y} = \frac{\mathrm{d}x}{x}\,.$$

  We now integrate both sides.

$$\int \frac{\mathrm{d}x}{x} = \log|x| + C$$

  while

$$\begin{aligned}\int \frac{\mathrm{d}y}{y(y-1)} &= \int \left(\frac{1}{y-1} - \frac{1}{y}\right)\mathrm{d}y = \log|y-1| - \log|y| + D \\ &= \log\left|\frac{y-1}{y}\right| + D\,.\end{aligned}$$

  We conclue that

$$\log|x| + C = \log\left|\frac{y-1}{y}\right| + D\,.$$

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Exponentiating both sides we get:

$$|x|\, e^{C-D} = \left|\frac{y-1}{y}\right| .$$

Writing $A = \pm e^{C-D}$so that the signs come out right we get

$$Ax = \frac{y-1}{y} = 1 - \frac{1}{y}$$

so that

$$\frac{1}{y} = 1 - Ax$$

and

$$y = \frac{1}{1-Ax} .$$

We need the solution such that $y(1) = -1$ that is such that $-1 = \frac{1}{1-A}$ which means $1 - A = -1$ so $A = 2$ and the solution is

$$\boxed{y = \frac{1}{1-2x} .}$$

(4) A physical system satisfies the equation $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E$. There m, k, E are constants (mass, spring constant, energy, respectively) and $v = \frac{\mathrm{d}x}{\mathrm{d}t}$ is the velocity.

(a) Solve the equation to obtain $\frac{\mathrm{d}x}{\mathrm{d}t} = v =$

Solution: $v = \sqrt{\frac{2E}{m} - \frac{k}{m}x^2}$.

(b) Suppose $m = k = 1$ and $E = \frac{1}{2}$. Integrate both sides of $\frac{\mathrm{d}x}{\sqrt{1-x^2}} = \mathrm{d}t$ and find a formula for $x = x(t)$.

(c) Solve the problem for general m, k, E.