Math 101 – SOLUTIONS TO WORKSHEET 15
PARTIAL FRACTIONS, APPROXIMATE INTEGRATION

1. Partial fractions expansion

(1) Write the form of the partial fraction expansion: $\frac{x^2+3x+6}{x^3(2x+3)(x-1)^2} =$

Solution: The degree of the numerator is smaller than that of the denominator, which is factored into powers of linear factors, so we get

$$\boxed{\frac{A}{x^3} + \frac{B}{x^2} + \frac{C}{x} + \frac{D}{2x+3} + \frac{E}{(x-1)^2} + \frac{F}{x-1}\,.}$$

(2) (Division)

(a) Factor $x^3 - 6x^2 + 11x - 6$

Solution: By the rational root theorem, rational roots might be $\pm1, \pm2, \pm3, \pm6$. We try: $1 - 6 + 11 - 6 = 0$ so 1 is a root and $(x-1)$ is a divisor. We divide by $x - 1$:

$$\begin{aligned} x^3 - 6x^2 + 11x - 6 &= x^2(x-1) - 5x^2 + 11x - 6 \\ &= x^2(x-1) - 5x(x-1) + 6x - 6 \\ &= (x^2 - 5x + 6)(x-1) \\ &= (x-2)(x-3)(x-1)\,. \end{aligned}$$

(b) Divide with remainder: $\frac{2x^3}{(x+2)(2x+3)} =$

Solution: We have $(x+2)(2x+3) = 2x^2 + 7x + 6$ so

$$\begin{aligned} 2x^3 &= x\left(2x^2 + 7x + 6\right) - 7x^2 - 6x \\ &= x\left(2x^2 + 7x + 6\right) - \frac{7}{2}\left(2x^2 + 7x + 6\right) - 18\frac{1}{2}x + 21 \\ &= \left(x - \frac{7}{2}\right)\left(2x^2 + 7x + 6\right) - \frac{37}{2}x + 21 \end{aligned}$$

and

$$\boxed{\frac{2x^3}{(x+2)(2x+3)} = \frac{21 - \frac{37}{2}x}{(x+2)(2x+3)} + \left(x - \frac{7}{2}\right)\,.}$$

(3) Consider $\frac{8x-10}{4x^3-4x^2+5x} = \frac{8x-10}{x(4x^2-4x+5)} = \frac{A}{x} + \frac{Bx+C}{4x^2-4x+5}$

(a) Find A using method 2

Solution: We have $\frac{8x-10}{x(4x^2-4x+5)} \sim_0 \frac{-10}{x(5)} = \frac{-2}{x}$ so $A = -2$.

(b) Calculate $\frac{8x-10}{x(4x^2-4x+5)} - \frac{A}{x}$ to find B, C.

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Solution: We have

$$\begin{aligned}
\frac{8x-10}{x(4x^2-4x+5)} - \frac{(-2)}{x} &= \frac{1}{x}\left[\frac{8x-10}{4x^2-4x+5}+2\right] \\
&= \frac{1}{x}\left[\frac{8x-10+2\left(4x^2-4x+5\right)}{4x^2-4x+5}\right] \\
&= \frac{1}{x}\left[\frac{8x^2+8x-8x-10+10}{4x^2-4x+5}\right] \\
&= \frac{8x^2}{x(4x^2-4x+5)} \\
&= \frac{8x}{4x^2-4x+5}
\end{aligned}$$

so that $B = 8$ and $C = 0$.

2. Numerical integration

(1) (Final 2009) Let $f(x) = \sin(e^x)$. Approximate $I = \int_0^2 f(x)\,\mathrm{d}x$ using the modpoint rule, the trapezoid rule, and Simpson's rule, with $n = 4$ in all cases. You may leave your answers in calculator-ready form.

Solution: We have $\Delta x = \frac{2}{4} = \frac{1}{2}$ so the points are $0, \frac{1}{2}, 1, \frac{3}{2}, 2$. The midpoint rule gives:

$$I \approx \frac{1}{2}\left[\sin\left(e^{1/4}\right) + \sin\left(e^{3/4}\right) + \sin\left(e^{5/4}\right) + \sin\left(e^{7/4}\right)\right]$$

Trapezoid:

$$I \approx \frac{1}{2}\left[\frac{1}{2}\sin\left(e^{0}\right) + \sin\left(e^{1/2}\right) + \sin\left(e^{1}\right) + \sin\left(e^{3/2}\right) + \frac{1}{2}\sin\left(e^{2}\right)\right]$$

Simpson:

$$I \approx \frac{1}{3\cdot 2}\left[\sin\left(e^{0}\right) + 4\sin\left(e^{1/2}\right) + 2\sin\left(e^{1}\right) + 4\sin\left(e^{3/2}\right) + \sin\left(e^{2}\right)\right]$$

(2) (Final 2015) Write down the Simpson's rule approximation to $I = \int_0^2 (x-3)^5\,\mathrm{d}x$ with $n = 6$.You may leave your answers in calculator-ready form.

Solution: In each case $\Delta x = \frac{2}{6} = \frac{1}{3}$. The solution is therefore

$$I \approx \frac{1}{3\cdot 3}\left[(0-3)^5 + 4\left(\frac{1}{3}-3\right)^5 + 2\left(\frac{2}{3}-3\right)^5 + 4\,(1-3)^5 + 2\left(\frac{4}{3}-3\right)^5 + 4\left(\frac{5}{3}-3\right)^5 + (2-3)^5\right].$$