Math 101 – SOLUTIONS TO WORKSHEET 13
INTEGRATION USING PARTIAL FRACTIONS

1. Tail end of Trig Substitution

(1) (105 Final, 2014 + 101 Final, 2009) Convert $\int (3-2x-x^2)^{-3/2}\,\mathrm{d}x$ to a trigonometric integral.

Solution: We complete the square: $3-2x-x^2 = 3+1-(1+2x+x^2) = 4-(x+1)^2$. So if we set $x+1 = 2\sin\theta$ we'd have $4-(x+1)^2 = 4-4\sin^2\theta = 4\cos^2\theta$. Since $x = 1+2\sin\theta$ we have $\mathrm{d}x = 2\cos\theta$ and we get

$$\begin{aligned}\int (3-2x-x^2)^{-3/2}\,\mathrm{d}x &= \int \left(4-4\sin^2\theta\right)^{-3/2} 2\cos\theta\,\mathrm{d}\theta \\ &= \frac{2}{4^{3/2}}\int \left(\cos^2\theta\right)^{-3/2}\cos\theta\,\mathrm{d}\theta \\ &= \frac{1}{4}\int \sec^2\theta\,\mathrm{d}\theta\end{aligned}$$

2. Partial fractions: Preliminaries

(1) (Polynomials)

(a) Which of the following is irreducible? x^2+7, x^2-7, $2x^2+3x-4$, $2x^2+3x+4$.

Solution: Recall that ax^2+bx+c is reducible iff $\Delta = b^2-4ac \geq 0$, so x^2+7, $2x^2+3x-4$ are reducible.

(b) Factor the polynomials x^2-3x+2, x^3-4x.

Solution: $x^2-3x+2 = (x-1)(x-2)$, $x^3-4x = x(x^2-4) = x(x-2)(x+2)$.

(2) (Preliminaries 2) Evaluate

(a) $\int \frac{\mathrm{d}x}{3x+4} =$

Solution: Let $u = 3x+4$, $\mathrm{d}u = 3\,\mathrm{d}x$. We have $\int \frac{\mathrm{d}x}{3x+4} = \frac{1}{3}\int\frac{\mathrm{d}u}{u} = \frac{1}{3}\log|u|+C = \frac{1}{3}\log|3x+4|+C$.

Solution: This is $\frac{1}{3}\int\frac{\mathrm{d}x}{x+4/3} = \frac{1}{3}\log\left|x+\frac{4}{3}\right|+C$.

PUZZLE: can you reconicle the seemingly distinct answers?

(b) $\int \frac{\mathrm{d}x}{(3x+4)^3} =$

Solution: Let $u = 3x+4$, $\mathrm{d}u = 3\,\mathrm{d}x$. We have $\int\frac{\mathrm{d}x}{(3x+4)^2} = \frac{1}{3}\int\frac{\mathrm{d}u}{u^2} = \frac{1}{3u}+C = -\frac{1}{3(3x+4)}+C$.

(c) $\int \frac{8x}{4x^2-4x+5}\,\mathrm{d}x = \int\frac{8x}{((2x-1)^2+4)}\,\mathrm{d}x =$

Solution: Let $u = 4x^2-4x+5$. Then $\mathrm{d}u = 8x-4$. We therefore split our integral as:

$$\int\frac{8x}{4x^2-4x+5}\,\mathrm{d}x = \int\frac{8x-4+4}{4x^2-4x+5}\,\mathrm{d}x = \int\frac{(8x-4)\,\mathrm{d}x}{4x^2-4x+5} + \int\frac{4\,\mathrm{d}x}{((2x-1)^2+4)}\,.$$

In the first integral we substitute $u = 4x^2-4x+5$ as planned. We recognize the second as an instance of x^2+a^2 and substitute $2x-1 = 2\tan\theta$ so that $2\,\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$ and $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$. We therefore have

$$\begin{aligned}\int\frac{8x}{4x^2-4x+5}\,\mathrm{d}x &= \int\frac{\mathrm{d}u}{u} + 4\int\frac{\sec^2\theta\,\mathrm{d}\theta}{\left(4\tan^2\theta+4\right)} \\ &= \log|u| + \int\frac{\sec^2\theta\,\mathrm{d}\theta}{\sec^2\theta} = \log|u|+\theta+C \\ &= \log\left(4x^2-4x+5\right) + \arctan\left(x-\frac{1}{2}\right)+C\end{aligned}$$

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(we used here that $\tan\theta = \frac{2x-1}{2} = x - \frac{1}{2}$ and that $4x^2 - 4x + 5 = (2x-1)^2 + 4$ is everywhere positive).

Solution: We recognize the expression in the demoninator as $(2x-1)^2 + 2^2$ so we substitute $2x - 1 = 2\tan\theta$, $2\,\mathrm{d}x = 2\sec^2\theta\,\mathrm{d}\theta$. This means $x = \tan\theta + \frac{1}{2}$ and $\mathrm{d}x = \sec^2\theta\,\mathrm{d}\theta$ so:

$$\begin{aligned}\int \frac{8x}{4x^2-4x+5}\,\mathrm{d}x &= \int \frac{4\left(\tan\theta + \frac{1}{2}\right)}{(2\tan\theta)^2+4}\sec^2\theta\,\mathrm{d}\theta \\ &= 2\int\left(\tan\theta+\frac{1}{2}\right)\frac{\sec^2\theta}{\tan^2\theta+1}\,\mathrm{d}\theta \\ &= 2\int\left(\tan\theta+\frac{1}{2}\right)\mathrm{d}\theta \\ &= 2\int\frac{\sin\theta\,\mathrm{d}\theta}{\cos\theta} + \int \mathrm{d}\theta \\ &= -2\log|\cos\theta| + \frac{1}{2}\theta + C \\ &= -\log\left|\cos^2\theta\right| + \frac{1}{2}\theta + C \\ &= -\log\cos^2\theta + \frac{1}{2}\theta + C \\ &= \log\sec^2\theta + \frac{1}{2}\theta + C\end{aligned}$$

(where we used $\sec\theta = \frac{1}{\cos\theta}$). Now $\tan\theta = x - \frac{1}{2}$ means $\theta = \arctan\left(x - \frac{1}{2}\right)$ and $\sec^2\theta = \tan^2\theta + 1 = (x-\frac{1}{2})^2 + 1 = x^2 - x + \frac{5}{4}$ and hence

$$\int \frac{8x}{4x^2-4x+5}\,\mathrm{d}x = \log\left(x^2 - x + \frac{5}{4}\right) + \frac{1}{2}\arctan\left(x - \frac{1}{2}\right) + C\,.$$

Bonus: note that $\log\left(4x^2-4x+5\right) = \log\left(4\left(x^2-x+\frac{5}{4}\right)\right) = \log\left(x^2-x+\frac{5}{4}\right) + \log 4$ so the two solutions indeed differ by a constant.

3. Partial fractions expansion

(1) Find A, B such that $\frac{5x+3}{(x+2)(2x-3)} = \frac{A}{x+2} + \frac{B}{2x-3}$:

- Clear denominators to get $5x + 3 =$
- (Method 1) Simplify and solve for A, B.

Solution: Clear denominators to get $5x+3 = A(2x-3) + B(x+2)$, simplify to get

$$5x + 3 = (2A+B)x + (2B-3A)$$

and hence the system of equations

$$\begin{cases} 2A + B &= 5 \\ -3A + 2B &= 3 \end{cases}.$$

Subtacting the second equation from twice the first gives $7A = 7$ so $A = 1$ and then $B = 3$. We verify the solution: $\frac{1}{x+2} + \frac{3}{2x-3} = \frac{2x-3+3(x+2)}{(x+2)(2x-3)} = \frac{5x+3}{(x+2)(2x-3)}$.

(2) Apply Method 2 to find A, B, C such that $\frac{6x^2-26x+26}{x^3-6x^2+11x-6} = \frac{6x^2-26x+26}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}$.

Solution: We have

$$\begin{aligned}\frac{6x^2-26x+26}{(x-1)(x-2)(x-3)} &\sim_1 \frac{6\cdot 1^2 - 26\cdot 1 + 26}{(x-1)(1-2)(1-3)} = \frac{6}{(x-1)(-1)(-2)} = \frac{3}{x-1} \\ \frac{6x^2-26x+26}{(x-1)(x-2)(x-3)} &\sim_2 \frac{6\cdot 2^2 - 26\cdot 2 + 26}{(2-1)(x-2)(2-3)} = \frac{-2}{(-1)(x-2)} = \frac{2}{x-1} \\ \frac{6x^2-26x+26}{(x-1)(x-2)(x-3)} &\sim_3 \frac{6\cdot 3^2 - 26\cdot 3 + 26}{(3-1)(3-2)(x-3)} = \frac{2}{(2)(x-2)} = \frac{1}{x-1}\end{aligned}$$

so $A = 3$, $B = 2$, $C = 1$.

(3) Now consider $\frac{8x-10}{4x^3-4x^2+5x} = \frac{8x-10}{x(4x^2-4x+5)} = \frac{A}{x} + \frac{Bx+C}{4x^2-4x+5}$

(a) Find A using method 2

Solution: We have $\frac{8x-10}{x(4x^2-4x+5)} \sim_0 \frac{-10}{x(5)} = \frac{-2}{x}$ so $A = -2$.

(b) Calculate $\frac{8x-10}{x(4x^2-4x+5)} - \frac{A}{x}$ to find B, C.

Solution: We have

$$\begin{aligned}
\frac{8x-10}{x(4x^2-4x+5)} - \frac{(-2)}{x} &= \frac{1}{x}\left[\frac{8x-10}{4x^2-4x+5} + 2\right] \\
&= \frac{1}{x}\left[\frac{8x-10+2\left(4x^2-4x+5\right)}{4x^2-4x+5}\right] \\
&= \frac{1}{x}\left[\frac{8x^2+8x-8x-10+10}{4x^2-4x+5}\right] \\
&= \frac{8x^2}{x(4x^2-4x+5)} \\
&= \frac{8x}{4x^2-4x+5}
\end{aligned}$$

so that $B = 8$ and $C = 0$.

(4) Finally consider $\frac{x^2}{(x+2)(2x-3)}$. Can we have A, B such that $x^2 = A(x+2) + B(2x-3)$?

Solution: No, because the degrees don't match.