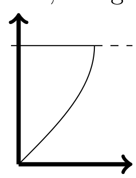
Math 101 – SOLUTIONS TO WORKSHEET 8
AREA BETWEEN CURVES, VOLUMES

(1) Find the total area of the following planar regions. It will be useful to sketch the region first.

(a) The finite region bounded by the y-axis, the graph of $y = \arcsin(x)$ and the line $y = \frac{\pi}{2}$.

Solution: We draw a sketch first.



Slicing vertically, requires evaluating

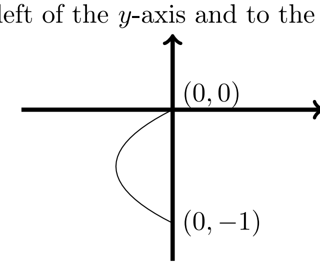
$$\int_{x=0}^{x=1} (1 - \arcsin x)\,\mathrm{d}x$$

which is painful. Slicing horizontally instead, we have $0 \le y \le \frac{\pi}{2}$ and at each y the length of the slice is $x = \sin y$ so instead we compute

$$\int_{y=0}^{y=\pi/2} \sin y\,\mathrm{d}y = [-\cos y]_{y=0}^{y=\pi/2} = 1\,.$$

(b) (Quiz, 2015) The finite region to the left of the y-axis and to the right of the curve $x = y^2 + y$.

Solution: We draw a sketch first.



Slicing horizontally, we have $-1 \le y \le 0$ and for each y, the slice has length $-(y^2 + y)$. The area is therefore

$$\int_{y=-1}^{y=0} (-y^2 - y)\,\mathrm{d}y = -\left[\frac{y^3}{3} + \frac{y^2}{2}\right]_{y=-1}^{y=0} = \frac{(-1)^3}{3} + \frac{(-1)^2}{2} = \boxed{\frac{1}{6}}.$$

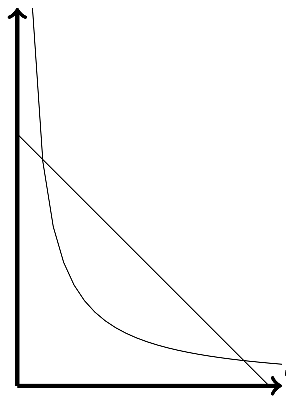
(2) Solids of revolution

(a) The area between the x-axis, the curve $y = x^2$ and the line $x = 5$ is revolved about the x-axis. What is the volume of the resulting region?

Solution: The volume is $\int_{x=0}^{x=5} \pi y^2\,\mathrm{d}x = \int_{x=0}^{x=5} \pi x^4\,\mathrm{d}x = \pi\left[\frac{x^5}{5}\right]_{x=0}^{5} = 5^4\pi = 625\pi$.

(b) (Final, 2014) Find the volume of the solid generated by rotating the finite region bounded by $y = \frac{1}{x}$ and $3x + 3y = 10$ about the x-axis. It will be useful to sketch the region first.

Date: 20/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.



Solution: The intersection points are where $x + \frac{1}{x} = \frac{10}{3}$ that is where $x^2 - \frac{10}{3}x + 1 = 0$ that is where $x = \frac{10/3 \pm \sqrt{\frac{100}{9} - 4}}{2} = \frac{10 \pm \sqrt{64}}{6} = \frac{5 \pm 4}{3} = \frac{1}{3}, 3$. The volume is therefore

$$\begin{aligned}
\pi \int_{x=1/3}^{x=3} \left(\left(\frac{10}{3} - x \right)^2 - \left(\frac{1}{x} \right)^2 \right) \mathrm{d}x &= \pi \int_{x=1/3}^{x=3} \left(\frac{100}{9} - \frac{20}{3}x + x^2 - x^{-2} \right) \mathrm{d}x \\
&= \pi \left[\frac{100}{9}x - \frac{10}{3}x^2 + \frac{x^3}{3} + \frac{1}{x} \right]_{x=1/3}^{x=3} \\
&= \pi \left[\left(300 - 90 + 9 + \frac{1}{3} \right) - \left(\frac{100}{27} - \frac{10}{27} + \frac{1}{81} + 3 \right) \right] \\
&= \pi \left[275\frac{17}{81} \right] = 275\frac{17}{81} \cdot \pi\,.
\end{aligned}$$