32. Manipulating power series 2 (30/3/2017)

Goals:

(1) Integration and differentiation of power series
(2) Finding power series expansions of functions and formulas for sums of power series.

Last time: Manipulating $\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$

Eg. $\frac{x^3}{1-x} = x^3 \cdot \frac{1}{1-x} = x^3(1+x+x^2 \ldots) = x^3+x^4+x^5+\ldots$

$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = 1 - x^3 + x^6 - x^9 + x^{12} - x^{15} + \ldots$

Problem: Power series expansion of $\frac{1}{7+x}$

(1) about $c=0$: $\frac{1}{7+x} = \frac{1}{7} \cdot \frac{1}{1+\frac{x}{7}} \underset{u=-\frac{x}{7}}{=} \frac{1}{7}\frac{1}{1-u} = \frac{1}{7}(1+u+u^2+u^3+\ldots)$

$= \frac{1}{7}\left(1 - \frac{x}{7} + \frac{x^2}{49} - \frac{x^3}{343} + \ldots\right) = \frac{1}{7} - \frac{x}{49} + \frac{x^2}{343} - \ldots$

$= \frac{1}{7}\sum_{n=0}^{\infty}\left(-\frac{x}{7}\right)^n = \sum_{n=0}^{\infty}(-1)^n \cdot \frac{1}{7^{n+1}} x^n$

(2) About $c=-3$

Solution will look like $A_0 + A_1(x+3) + A_2(x+3)^2 + \ldots$ So need $x+3$:

$\frac{1}{7+x} = \frac{1}{4+x+3} = \frac{1}{4} \cdot \frac{1}{1+\frac{x+3}{4}} \underset{u=-\frac{x+3}{4}}{=} \frac{1}{4}\sum_{n=0}^{\infty}\left(-\frac{x+3}{4}\right)^n$

want $\frac{1}{1-u}$

$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n \cdot \left(\frac{1}{4}\right)^n \cdot (x+3)^n = \sum_{n=0}^{\infty}(-1)^n \cdot \left(\frac{1}{4}\right)^{n+1} (x+3)^n$

Math 101 – WORKSHEET 32
MANIPULATING POWER SERIES

1. Manipulating power series: Calculus

(1) Let $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$. We know that f converges everywhere, while g converges in $(-1,1]$.

(a) Find the power series representation of $f'(x)$. What is $f(x)$?

$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, $f'(x) = \sum_{n=0}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{m=0}^{\infty} \frac{x^m}{m!} = f(x)$

$\frac{d}{dx}1 = 0$ $\quad n! = n\cdot(n-1)!$ $\quad n-1=m$

So $f' = f$, so $f(x) = C\cdot e^x$,

need $f(0) = \frac{1}{0!} = 1$ so $C = 1$, $f(x) = e^x$

(b) Find the power series representation of $g'(x)$. What is $g'(x)$? What is $g(x)$?

$g'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cdot n \cdot x^{n-1} = \sum_{n=1}^{\infty} (-1)^n \cdot x^{n-1} = \sum_{m=0}^{\infty} (-1)^{m+1} \cdot x^m = -\sum_{m=0}^{\infty} (-x)^m$

$n-1=m$

$= -\frac{1}{1-(-x)} = -\frac{1}{1+x}$

i.e. $g'(x) = -\frac{1}{1+x}$, $g(x) = -\log(1+x)$

(1) found $g'(x)$
(2) noticed $g'(x)$ is a geometric series
(3) summed that series
(4) found $g(x)$
(5) Get radius of $g(x)$ is 1

(c) Conclude that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2$.

plug in $x=1$, get $-\log(2) = g(1) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

so $\log(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

Date: 27/3/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Fact: If power series $\sum_{n=0}^{\infty} A_n(x-c)^n$ has radius of convergence R, then the derivative is $\sum_{n=0}^{\infty} nA_n(x-c)^{n-1}$

indefinite integral is $C' + \sum_{n=0}^{\infty} A_n \frac{1}{n+1}(x-c)^{n+1}$

with same radius of convergence. (but endpoints can differ)

More on 1(a): $f(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)$

$$f'(x) = \left(\cancel{0} + 1 + x + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \dots\right)$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Skill: Given a power series, find derivative

note: need to shift variable n by 1.

Extra example: Find expansion of arctan(x) about $c=0$.

Let $f(x) = \arctan x$, then $f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$

$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n$

i.e. $f'(x) = \sum_{n=0}^{\infty} (-1)^n \cdot x^{2n}$

so $f(x) = C' + \sum_{n=0}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$ but $\arctan(0) = 0$ so $C' = 0$

so $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ $\Rightarrow$ $\frac{\pi}{4} = \arctan(1) = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \dots$

(2) Consider the *error function* $\text{erf}(x) = \int_0^x \exp(-t^2)\, dt$.

(a) Find the power series expansion of $\text{erf}(x)$ about zero.

Recall: $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ so $\exp(-t^2) = \sum_{n=0}^{\infty} \frac{1}{n!}(-t^2)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot t^{2n}$

So $\int_0^x e^{-t^2} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^x t^{2n} dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \left[\frac{t^{2n+1}}{2n+1}\right]_0^x = \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \cdot (2n+1)} \cdot x^{2n+1}$

can integrate term-by-term

(b) How many terms in the expansion are necessary to estimate $\text{erf}(\frac{1}{2})$ to within 0.001?

- Apply AST

Short Table of standard expansions

You must either memorize the following expansions or be able to quickly reproduce them.

- (geometric series)

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

- (Exponential)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

- (Trig)

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \end{aligned}$$

- (logarithm)

$$\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$$