Math 101, lecture 18:

Goals: (1) Comparison of integrals

(2) Work

Last time: Integral is improper if (1) domain infinite
(2) integrand is unbounded

Handle by introducing cutoff, shifting the cutoff:

$$\int_1^\infty \frac{dx}{x^p} = \lim_{T\to\infty} \int_1^T \frac{dx}{x^p} \leftarrow \text{converse if } p>1$$

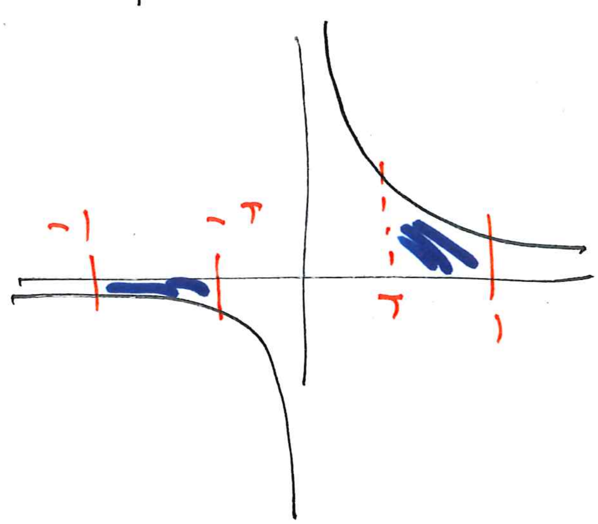
$$\int_0^1 \frac{dx}{x^p} = \lim_{T\to 0} \int_T^1 \frac{dx}{x^p} \leftarrow \text{converge if } p<1$$

If multiple bad points, split domain: $\int_0^\infty \frac{dx}{x^p} = \int_0^1 + \int_1^\infty$

(this one diverges for all p)

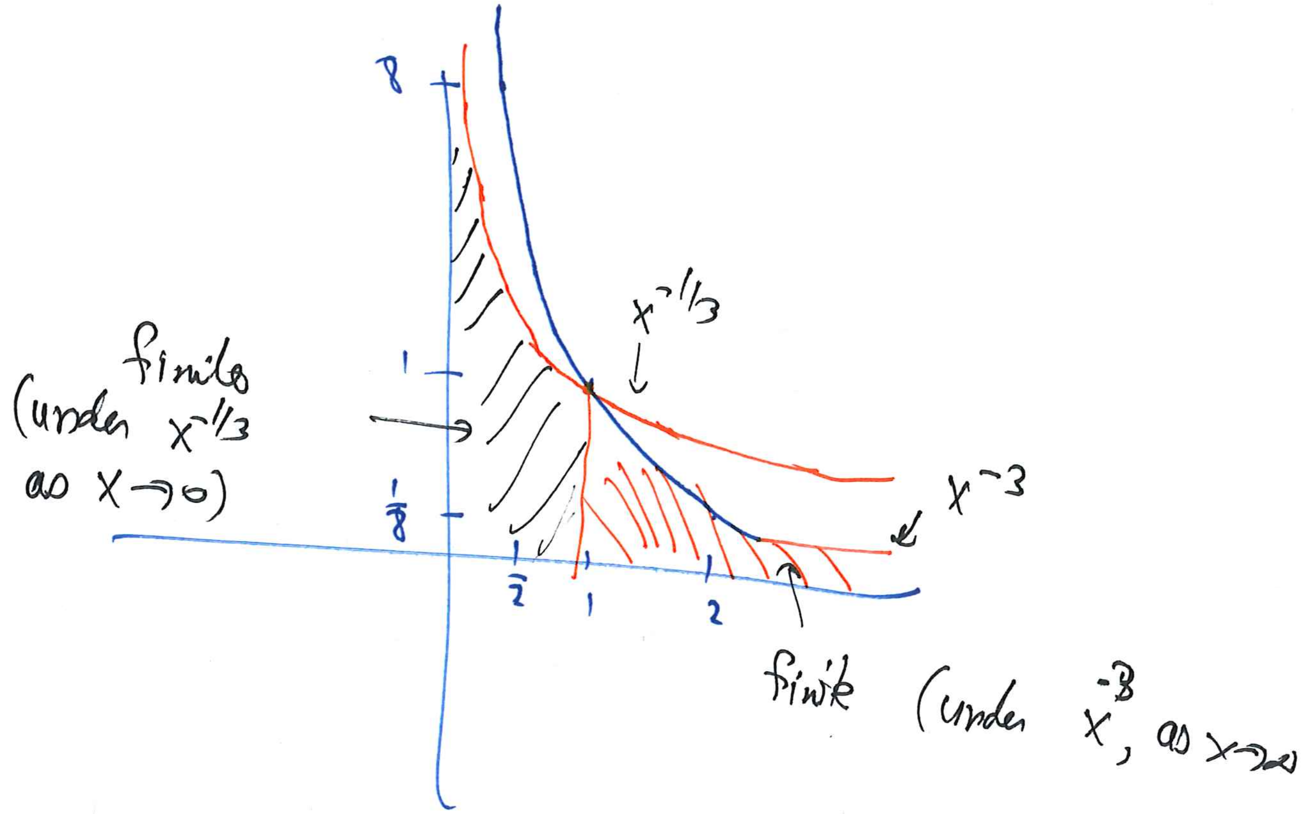
To study $\int_{-1}^1 \frac{dx}{x^3}$ need to study $\int_{-1}^0 \frac{dx}{x^3}$, $\int_0^1 \frac{dx}{x^3}$ separately.

Both diverge (p-integrals $p\geq 1$), so integral diverges



but $\int_{-1}^1 x^{-1/3}\,dx = 0$

(integral exists on both sides: $(p=\frac{1}{3}<1)$
so can use symmetry.



Fact: If an area is smaller than a finite area, it is finite.
If an area is larger than an infinite area, it is infinite.
(larger than finite, or smaller than infinite is undetermined)

Note: $\int_e^\infty \frac{dx}{x(\log x)^p}$ can converge or diverge!
($p>0$ so $\frac{1}{x(\log x)^p} < \frac{1}{x}$)

Math 101 – WORKSHEET 18
IMPROPER INTEGRALS, WORK

1. Comparison of Integrals

(1) Decide which of the following integrals converge

(a) (103 Final, 2012) $\int_1^\infty \frac{1+\sin x}{x^2}\,dx$.

[thought: this decays like $\frac{1}{x^2}$, so should converge] Have $0 \le 1+\sin x \le 2$

so $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$, and $\int_1^\infty \frac{2}{x^2}\,dx = 2\int_1^\infty \frac{dx}{x^2}$ converges ($p=2>1$)

check positivity

make upper bd.

so $\int_1^\infty \frac{1+\sin x}{x^2}\,dx$ converges by comparison

make comparison

(b) $\int_1^\infty \frac{3-\cos x}{x}\,dx$.

$2 \le 3-\cos x \le 4$ so $\frac{3-\cos x}{x} \ge \frac{2}{x}$, $\int_1^\infty \frac{2}{x}\,dx = \infty$ (p-integral, $p=1$) so by comparison $\int_1^\infty \frac{3-\cos x}{x}\,dx$ diverges too

(c) (Bell curve) $\int_{-\infty}^{+\infty} e^{-x^2}\,dx$

(d) $\int_0^1 \frac{dx}{\sqrt{x}+\sin x}$

(e) (hard) $\int_0^1 \frac{dx}{x^2+x^3}$

(f) (hard) $\int_0^\infty \frac{x^{1000}}{e^x}\,dx$

Date: 15/2/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

Work

Definition: A force F acts on a body along distance Δx, it does work $F\Delta x$ on the body.

(Pushing against constant gravity, mass m corresponds force mg, $g \approx 9.8 \frac{m}{sec^2}$ acceleration - on Earth)

Worksheet (1), (2), (3)

Summary: To calculate work on pointlike object by variable force, chop up the process, i.e. the distance increments.

2. WORK

(1) (Preliminary) A worker carries a 20kg bucket to the top of a 10m tall building. Half way up the worker picks up a second 20kg bucket. Calculate the total work done by the worker: by adding the contributions from carrying each bucket separately.

First, carry 20Kg up 5m, work is $20\text{Kg} \cdot 9.8 \frac{m}{sec^2} \cdot 5m = 980\ J$

Then carry 40Kg up 5m, work is double, 1960 J, total is 2940 J

unit of work: $1\ \text{Kg} \cdot \frac{m^2}{sec^2} = 1$ Joule $= 1$ J.

(2) When a spring is displaced $x$cm from its equillibrium position it exerts a force of $5x$ Newtons (i.e the force is $F(x) = 5x$). Find the work required to stretch the spring from a displacement of 20cm to a displacement of 60cm.

To move spring from displacement x to $x+dx$ requires work:

$F(x) \cdot dx$

So total work is $\boxed{\int_{x_{initial}}^{x_{final}} F(x)\,dx}$

Here, it is $\int_{0.2}^{0.6} 5x\,dx$

(3) According to Newton's universal law of gravitation, the force between a planet of mass M and a probe of mass m is $F = \frac{GMm}{r^2}$ where r is the distances between them and $G \approx 6.67 \cdot 10^{-11}\mathrm{m}^3\mathrm{kg}^{-1}\mathrm{s}$ is the gravitational constant. Find the work required to lanuch a probe from the surface of the planet (radius R) all the way to infinity.

Work: $\int \text{force } d(\text{distance}) = \int_{r=R}^{r=\infty} \frac{GMm}{r^2}\, dr = GMm \int_{r=R}^{\infty} \frac{dr}{r^2}$

$\int_R^\infty \frac{dr}{r^2} = \lim_{T\to\infty} \int_R^T \frac{dr}{r^2} = \lim_{T\to\infty} \left[-\frac{1}{r}\right]_R^T = \lim_{T\to\infty} \left(\frac{1}{R} - \frac{1}{T}\right) = \frac{1}{R}$ so

$\boxed{\text{work} = \frac{GMm}{R}}$

(4) In the Morse model for a diatomic molecule (e.g. H_2, O_2 etc), when the two atoms are separated by distance x, the force between them is

$$F(x) = 2E\left(1 - e^{-(x-r)}\right) e^{-(x-r)}$$

where r is the separation between the atoms at the equillibrium position and E is a parameter. Find the work required to dissociate the molecule, by taking an atom all the way from separation r to ∞.