13. Partial Fractions I (1/2/2017)

Goals.

(1) Little bit more on trig substitution
(2) Partial fractions
  (a) Why ???
  (b) How to compute the expansion
  (c) How to use

Last time: Trig substitutions:

$a^2 - x^2$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$a^2 + x^2$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$x^2 - a^2$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

- Must converge back to x. Won't accept $\arcsin(\cos\theta)$ or $\cos(\arctan(x))$.

Q: Material for Quiz 2? A: Units 3,4 (different topics from last year)

Math 101 – WORKSHEET 13
INTEGRATION USING PARTIAL FRACTIONS

1. Tail end of Trig Substitution

(1) (105 Final, 2014 + 101 Final, 2009) Convert $\int \left(3 - 2x - x^2\right)^{-3/2} dx$ to a trigonometric integral.

Here, $3 - 2x - x^2 = 3 - (x^2 + 2x) = 3 - (x^2 + 2x + 1) + 1$

$= 4 - (x+1)^2$ ← complete the square

so problem is about

$$\int \frac{dx}{\left(4 - (x+1)^2\right)^{3/2}}$$, want to substitute

$x + 1 = 2\sin\theta$

$dx = 2\cos\theta\, d\theta$ so

$$\int \frac{2\cos\theta\, d\theta}{\sqrt{4 - 4\sin^2\theta}^3} d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sqrt{1 - \sin^2\theta}^3} d\theta = \frac{1}{4}\int \frac{d\theta}{\cos^2\theta}$$

Date: 1/2/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

2. Partial fractions: Preliminaries

(1) (Polynomials)

(a) Which of the following is irreducible? x^2+7, x^2-7, $2x^2+3x-4$, $2x^2+3x+4$.

$x^2-7=(x+\sqrt{7})(x-\sqrt{7})$. Know: in ax^2+bx+c, let $\Delta=b^2-4ac$, quadratic is reducible iff $\Delta\geq 0$

$2x^2+3x-4$: $\Delta=9+32>0$, $2x^2+3x+4$: $\Delta=9-32<0$

(b) Factor the polynomials x^2-3x+2, x^3-4x.

$x^2-3x+2=(x-1)(x-2)$, $x^3-4x=x(x^2-4)=x(x+2)(x-2)$

obvious factor ← → more

(2) (Preliminaries 2) Evaluate

(a) $\int \frac{dx}{3x+4} = \frac{1}{3}\log|3x+4|+C$

can substitute $u=3x+4$

(b) $\int \frac{dx}{(3x+4)^3} = -\frac{1}{6(3x+4)^2}+C$

want to put $u=(2x-1)^2$, $du=4(2x-1)dx$

(c) $\int \frac{8x}{4x^2-4x+5}\,dx = \int \frac{8x}{((2x-1)^2+4)}\,dx = \int \frac{8x-4}{(2x-1)^2+4}\,dx + \int \frac{4}{(2x-1)^2+4}\,dx$

$= \int \frac{2du}{u+4} + \int \frac{dx}{(x-\frac{1}{2})^2+1} = \log|u+4| + \arctan(x-\frac{1}{2})+C$

$= \log((2x-1)^2+4) + \arctan(x-\frac{1}{2})+C$

$u=(2x-1)^2$, $du=(8x-4)dx$

(or: $2x-1=2\tan\theta$)

$\frac{a^2}{4}=(\frac{a}{2})^2$

Partial fractions motivation

(1) Would you rather compute

$$\square \int \left(\frac{6x^2-22x+18}{x^3-6x^2+11x-6}\right)\,\mathrm{d}x$$

$$\square \int \left(\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}\right)\,\mathrm{d}x$$

(2) Compare

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \arctan x + C$$

with

$$\int \frac{2}{x^2-1}\,\mathrm{d}x = \log\left|\frac{x-1}{x+1}\right| + C$$

what is the difference?

(3) "Magic identities":

$$\frac{6x^2-22x+18}{x^3-6x^2+11x-6} = \frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}$$

and

$$\frac{2}{x^2-1} = \frac{1}{x-1}-\frac{1}{x+1}$$

Know in advance need piece blowing up at -2, at $3/2$

3. Partial fractions expansion

(1) Find A, B such that $\frac{5x+3}{(x+2)(2x-3)} = \frac{A}{x+2} + \frac{B}{2x-3}$:

- Clear denominators to get $5x + 3 = A(2x-3) + B(x+2)$
- (Method 1) Simplify and solve for A, B.

$5x+3 = 2Ax - 3A + Bx + 2B = (2A+B)x + (2B-3A)$

Want: $\begin{cases} 5 = 2A+B \\ 3 = 2B-3A \end{cases} \Rightarrow \begin{matrix} 10 = 4A+2B \\ 3 = -3A+2B \end{matrix} \Rightarrow 7 = 7A \Rightarrow \boxed{A=1}$

$\Rightarrow \boxed{B=3}$

i.e $\frac{5x+3}{(x+2)(2x-3)} = \frac{1}{x+2} + \frac{3}{2x-3}$

"Method 1": clear denominators, solve for coeff

(2) Apply Method 2 to find A, B, C such that

$$\frac{6x^2-26x+26}{x^3-6x^2+11x-6} = \frac{6x^2-26x+26}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

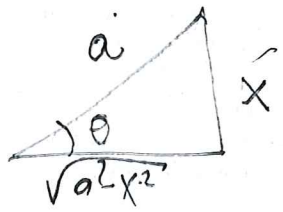
As $x \to 1$, $\frac{6x^2-26x+26}{(x-2)(x-3)} \to \frac{6-26+26}{(1-2)(1-3)} = 3 \Rightarrow \boxed{A=3}$

As $x \to 2$, $\frac{6x^2-26x+26}{(x-1)(x-3)} \to \frac{24-52+26}{(2-1)(2-3)} = 2$ $\boxed{B=2}$

As $x \to 3$ $\frac{6x^2-26x+26}{(x-1)(x-2)} \to \frac{6\cdot 9 - 26\cdot 3 + 26}{(3-1)(3-2)} = 1$ $\boxed{C=1}$

$X = a\sin\theta$ $\theta = \arcsin\left(\frac{X}{a}\right)$

$\cos\theta = ?$ $\sin\theta = \frac{X}{a}$



$\tan\theta$

$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-\left(\frac{X}{a}\right)^2}$