11. Trigonometric integrals (27/1/2017)

Goals.

(1) Two specific cases of integration

Summary of feedback (Thanks!)

- Most: keep lectures as they are. Minority: more or fewer examples.
  - Can't slow down pace of material.
- Room is too warm
- Webwork questions of variable difficulty
- Notes on the bottom of the screen

Last time: integration by parts.

Today: Method for evaluating $\int \sin^n x \cdot \cos^m x \, dx$

$$\int \tan^n x \cdot \sec^m x \, dx$$

Example 6: $\cancel{\int \cos^5 x \sin x}$ $\int \sin^4 x \cos x \, dx = \int u^4 \, du = \frac{u^5}{5} + C = \frac{\sin^5 x}{5} + C$

subst

$u = \sin x$

$du = \cos x \, dx$

$\boxed{(\cos x)' = -\sin x}$

$\cos^2 x = 1-\sin^2 x$

(1) Evaluate the integrals

(a) $\int \sin^4 x \cos^3 x \, dx = \int \sin^4 x \cdot \cos^2 x \cdot \cos x \, dx$

$= \int u^4 (1-u^2)\, du$

$u = \sin x$
$du = \cos x \, dx$

$= \int (u^4 - u^6)\, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

rule: This will work if power of $\cos x$ is odd.

(b) $\int \sin^5 x \cos^4 x \, dx = \int \sin^4 x \cdot \cos^4 x \cdot \sin x \, dx$

$= -\int (1-u^2)^2 \cdot u^4 \, du = -\int (1-2u^2+u^4) u^4 \, du =$

$u = \cos x$ | $\sin^4 x = (\sin^2 x)^2 = (1-u^2)^2$
$du - \sin x \, dx$ |

$= \int (2u^6 - u^4 - u^8)\, du = \frac{2u^7}{7} - \frac{u^5}{5} - \frac{u^9}{9} + C$

$= \frac{2}{7} \cos^7 x - \frac{1}{5}\cos^5 x - \frac{1}{9}\cos^9 x + C$

rule: use this when power of $\sin x$ is odd

Date: 27/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

$$\boxed{\cos^2 x = \frac{1+\cos(2x)}{2} \qquad \sin^2 x = \frac{1-\cos(2x)}{2}}$$

(c) $\int \sin^4 x \cos^4 x \, dx =$

$$= \int \left(\frac{1-\cos(2x)}{2}\right)^2 \left(\frac{1+\cos(2x)}{2}\right)^2 dx = \frac{1}{16}\int \left[(1-\cos(2x))(1+\cos(2x))\right]^2 dx$$

half-angle formulae

$$= \frac{1}{16}\int (1-\cos^2(2x))^2 = \frac{1}{16}\int (\sin^2(2x))^2 dx =$$

got $\sin^4(2x)$ use formula again

$$= \frac{1}{16}\int \left(\frac{1-\cos(4x)}{2}\right)^2 dx = \frac{1}{64}\int (1-2\cos(4x)+\cos^2(4x))\,dx$$

$$= \frac{x}{64} - \frac{1}{128}\sin(4x) + \frac{1}{64}\int \cos^2(4x)\,dx \quad ,,,,$$

rule: If powers of $\sin x$, $\cos x$ both even, use half-angle formulae.

(d) $\int \sin^5 x \cos^3 x \, dx = \int \sin^5 x \,(1-\sin^2 x)\cdot \cos x \, dx$

$$= \int u^5 (1-u^2)\,du = \frac{\sin^6 x}{6} - \frac{\sin^8 x}{8} + C$$

$u = \sin x$
$du = \cos x \, dx$

rule: If both powers are odd, either substitution is good

(2) Powers of tangent and secant

(a) Evaluate $\int_0^{\pi/4} \tan x\, dx$ $= \int_{x=0}^{x=\pi/4} \frac{\sin x}{\cos x} dx \underset{\uparrow}{=} \int_{u=1}^{u=1/\sqrt{2}} \frac{-du}{u} =$

$= \int_{u=1/\sqrt{2}}^{u=1} \frac{du}{u} = \left[\log|u|\right]_{1/\sqrt{2}}^{1} = \log 1 - \log \frac{1}{\sqrt{2}}$ $\quad u = \cos x, \; du = -\sin x\, dx$

$= \frac{1}{2}\log 2.$

(b) Evaluate $\int_{-\pi/4}^{+\pi/4} \tan x\, dx$ $= 0$

(odd function on a symmetric interval)

(c) (even power of secant) Evaluate $\int \tan^5 x \sec^4 x\, dx$ using the substitution $u = \tan x$, $du = \sec^2 x\, dx$ so:

$\int \tan^5 x \sec^4 x\, dx = \int u^5 (u^2+1)\, du = \int (u^7 + u^5\, du) = \frac{1}{8}\tan^8 x + \frac{1}{6}\tan^6 x + C$

$\sec^2 x = 1 + \tan^2 x$

$= \int \tan^5 x \cdot \sec^2 x\, (\sec^2 x\, dx)$

(d) (odd power of tangent) Write $\int \tan^5 x \sec^3 x\, dx$ in the form $\int \sin^n x \cos^m x\, dx$ and evaluate it.

(or: substitute $u = \sec x$)

$\int \tan^5 x \sec^3 x\, dx = \int \frac{\sin^5 x}{\cos^5 x} \cdot \frac{1}{\cos^3 x} dx = \int \frac{\sin^5 x}{\cos^8 x} dx \underset{\uparrow}{=} \int \frac{\sin^4 x}{\cos^8 x} \cdot \sin x\, dx =$

$\underset{u = \cos x\, dx}{=} -\int \frac{(1-u^2)^2}{u^8} du = \int \frac{2u^2 - 1 - u^4}{u^8} du = \int (2u^{-6} - u^{-8} - u^{-4})\, du =$

$= -\frac{2}{5}\cos^{-5} x + \frac{1}{7}\cos^{-7} x + \frac{1}{3}\cos^{-3} x + C$

$= -\frac{2}{5}\sec^5 x + \frac{1}{7}\sec^7 x + \frac{1}{3}\sec^3 x + C$

To integrate $\int \sin^n x \cos^m x \, dx$:

(1) If m is odd, use $u = \sin x$
(2) If n is odd, use $u = \cos x$
(3) If both even, use half-angle formulas

To integrate $\int \tan^n x \sec^m x \, dx$:

(1) If m is even, use $u = \tan x$
(2) If n is odd, use $u = \sec x$
(3) If m is odd, n even - not in 101

Or use $\tan x = \frac{\sin x}{\cos x}$, $\sec x = \frac{1}{\cos x}$ and rules above