# 6. Substitution (16/1/2017)

Goals.

(1) Immediate substitutions
(2) $u$-substitution
(3) Pitfalls

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Last Time: Indefinite integrals = anti-derivatives

Q: if $x_i = a + i\Delta x$ what's $x_{i-1}$? A: $x_{i-1} = a + (i-1)\Delta x$

Want to find anti-derivatives. Idea: Diff rules $\Rightarrow$ integration rules

Example: $(f+g)' = f' + g'$ $\Rightarrow$ $\int (f+g)\,dx = \int f\,dx + \int g\,dx$
$(af)' = af'$ $\Rightarrow$ $\int (af)\,dx = a\int f\,dx$

Today: chain rule.

Example: Say $F(x) = \int f(x)\,dx \iff F'(x) = f(x)$
Problem: find $\int f(ax)\,dx$

Solution 1: Think to try $F(ax)$: $\frac{d}{dx}\underbrace{(F(ax))}{} \overset{\text{chain rule}}{=} aF'(ax)$ so $\int f(ax)\,dx = \frac{1}{a}F(ax) + C$
$= af(ax)$

Solution 2: $\int f(ax)\,dx = \int f(ax)\frac{a}{a}\,dx = \frac{1}{a}\int \underbrace{f(ax)\,a}_{F'(ax)}\,dx = \frac{1}{a}F(ax) + C$

Example: $\int x e^{x^2}\,dx = \frac{1}{2}\int e^{x^2}\cdot 2x\,dx \overset{\text{chain rule}}{=} \frac{1}{2}e^{x^2} + C$

chain rule: $(f(g(x)))' = f'(g(x))\cdot g'(x)$

Name for this. Substitution.

Another way: $\int e^{x^2} x\,dx \underset{\substack{\uparrow \\ x^2 = u}}{=} \frac{1}{2}\int e^{u} (2x)\,dx = \frac{1}{2}\int e^{u} \frac{du}{dx}\cdot dx \underset{\substack{\uparrow \\ \frac{d(u)}{dx} = 2x}}{=} \frac{1}{2}\int e^{u}\,du =$

$= \frac{1}{2} e^{u} + C = \frac{1}{2} e^{x^2} + C$

Usual way: $\int e^{x^2} x\,dx \underset{\substack{\uparrow \\ u = x^2}}{=} \int e^{u} \frac{1}{2}\,du = \frac{1}{2} e^{u} + C \overset{\text{go back to } x}{=} \frac{1}{2} e^{x^2} + C$

$\frac{du}{dx} = 2x \iff du = 2x\,dx$

Summary: find piece $u = g(x)$ so that integrand is $f(u)\,du$
then integrate in $u$-variable
then return to $x$-variable

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Worksheet

Math 101 – WORKSHEET 6
SUBSTITUTION

**Theorem** (Substitution). $\int f'(g(x))g'(x)\,\mathrm{d}x = f(g(x)) + C$. *Equivalently,* $\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$ *where* $u = g(x)$.

(1) Evaluate the integrals

(a) $\int \sin x \cos x\,\mathrm{d}x =$

(hint: use $u = \sin x$)

aha - this is $(\sin x)'$ rest only depends on $\sin x$,

so let's try $u = \sin x$, $du = \cos x\,dx$ so

$$\int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C$$

**Problem.** It's easy to check that $\left(-\frac{1}{4}\cos(2x)\right)' = \frac{1}{2}\sin(2x) = \frac{1}{2}\cdot 2\sin x\cos x = \sin x\cos x$. How is that possible?

By half-angle formula, $\frac{1}{2}\sin^2 x$, $-\frac{1}{4}\cos(2x)$ differ by a constant.

$\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

so $-\frac{1}{4}\cos(2x) = -\frac{1}{4} + \frac{1}{2}\sin^2 x$

*Date*: 16/1/2017, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) (Final, 2014) $\int \cos^3 x \sin^4 x \, dx =$

Wrong: $\int \cos^3 x \sin^4 x = \int \cos^3 x\,(1-\cos^2 x)^2 dx = \int (\cos^3 x - 2\cos^5 x + \cos^7 x)\,dx$

$\neq \frac{\cos^4 x}{4} - \frac{2\cos^6 x}{6} + \frac{\cos^8 x}{8}$

$u = \sin x,\ du = \cos x\, dx$

Right: $\int \cos^2 x \sin^4 x \cdot (\cos x\, dx) = \int (1-\sin^2 x)\sin^4 x \cos x\, dx =$

$= \int (1-u^2)\cdot u^4\, du = \int (u^4 - u^6)\, du = \frac{u^5}{5} - \frac{u^7}{7} + C = \frac{\sin^5 x}{5} - \frac{\sin^7 x}{7} + C$

(c) (Final, 2013) $\int_1^3 (2x-1)e^{x^2-x}\, dx =$

(notice: $(x^2-x)' = 2x-1$)

$u = x^2 - x$

$du = 2x - 1$

$= \int_{x=1}^{x=3} e^u\, du = \int_{u=0}^{u=6} e^u\, du = \left[e^u\right]_{u=0}^{u=6} = e^6 - 1$

pitfall: plug in x-value in u or u-value in x

$= \int_{x=1}^{x=3} e^u du = \left[e^u\right]_{x=1}^{x=3} \overset{\text{plug into } u = x^2 - x}{=} \left[e^{x^2-x}\right]_{x=1}^{x=3} = e^6 - e^0 = e^6 - 1$

Method 3: compute indef integral first.

(d) (Final, 2012) $\int_0^3 (x+1)\sqrt{9-x^2}\, dx =$

$= \int_0^3 x\sqrt{9-x^2}\, dx + \int_0^3 \sqrt{9-x^2}\, dx = -\frac{1}{2}\int_{u=9}^{u=0} \sqrt{u}\, du + \int_0^3 \sqrt{9-x^2}\, dx$

$u = 9 - x^2$

$du = -2x$

area of semicircle