MATH 101: CONDITIONAL CONVERGENCE AND REARRANGING A SERIES

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This note illustrates ideas from Math 101, but its level of reasoning is beyond the level of the course and is NOT EXAMINABLE – the note is intended for you edification, not as an example of a worked problem.

In this note we'll see that rearranging a conditionally convergent series can change its sum. Along the way we'll:

- Understand series through their *partial sums*;
- See that *cancellation* is the cause of convergence of alternating series;
- Apply the *limit comparison test*, using *p-series* for comparison;
- Undersatnd a numerical series by extending it to a *power series*;
- Sum a power series by recognizing a *differential equation* it solves;
- Recognize an expression as a *Riemanmn sum,* and use that observation to evaluate a limit; and
- See that exactly evaluating sums of series usually requires some trickery.

Exercise. As you read this note, try to identify where each of the above techniques is used.

Remark. If this note "speaks to you" please come to my office where I'll be happy to point you toward further interesting mathematics.

1. The alternating harmonic series

We begin with the series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$.

1.1. The series converges.

Proof 1. The terms *alternate in sign*, *decrease in magnitude* and *tend to zero*, so the alternating series test applies. □

Proof 2. Since the terms go to zero, we can put parentheses as follows:

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots = \sum_{n=1}^{\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right) = \sum_{n=1}^{\infty}\left(\frac{2n-(2n-1)}{2n(2n-1)}\right)$$

$$= = \sum_{n=1}^{\infty}\frac{1}{2n(2n-1)}\,.$$

Now $\frac{1}{2n(2n-1)} = \frac{1}{4n^2}\left(\frac{1}{1-\frac{1}{2n}}\right)$ so the ratio $\frac{1}{2n(2n-1)}\Big/\frac{1}{n^2} \xrightarrow[n\to\infty]{} \frac{1}{4}$ and by the limit comparison test the series converges ($\sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series). □

1.2. Its sum is $\log 2$.

We deform our series to the power series $f(x) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$. Since $\left|\frac{(-1)^{n-1}x^n}{n}\right| \leq |x|^n$ and since $\sum_{n=1}^{\infty}|x|^n$ converges for $|x| < 1$ (geometric series), our series converges absolutely for $|x| < 1$. In the region of absolute convergence we have $f'(x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1} = \sum_{n=0}^{\infty}(-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$ so integration gives $f(x) = \log(1+x) + C$. Since $0 = f(0) = C$ we have for $-1 < x < 1$ that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n = \log(1+x)\,.$$

This note is specifically excluded from the terms of UBC Policy 81.

We have also seen that the series $f(1) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges. By Abel's Theorem we then have

$$f(1) = \lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = \lim_{x\to 1^-} \log(1+x) = \log 2\,.$$

2. Rearranging the series

Consider instead the series obtained by taking two odd terms followed by an even term:

$$\left(1+\frac{1}{3}-\frac{1}{2}\right)+\left(\frac{1}{5}+\frac{1}{7}-\frac{1}{4}\right)+\left(\frac{1}{9}+\frac{1}{11}-\frac{1}{6}\right)+\left(\frac{1}{13}+\frac{1}{15}-\frac{1}{8}\right)+\cdots \quad = \\ 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\frac{1}{8}+\cdots$$

2.1. **Convergence.** The new series is no longer alternating, so we adapt Proof 2 from above. For this we parametrize our series as

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right) &= \sum_{n=1}^{\infty} \frac{2n(4n-1)+2n(4n-3)-(4n-1)(4n-3)}{2n(4n-3)(4n-1)}\,. \\ &= \sum_{n=1}^{\infty} \frac{8n^2-2n+8n^2-6n-16n^2+16n-3}{32n^3\left(1-\frac{3}{4n}\right)\left(1-\frac{1}{4n}\right)} \\ &= = \sum_{n=1}^{\infty} \frac{8n-3}{32n^3\left(1-\frac{3}{4n}\right)\left(1-\frac{1}{4n}\right)} = \frac{1}{4}\sum_{n=1}^{\infty} \frac{\left(1-\frac{3}{8n}\right)}{\left(1-\frac{3}{4n}\right)\left(1-\frac{1}{4n}\right)}\frac{1}{n^2}\end{aligned}$$

which converges by the limit comparison test (compare with $\sum_{n=1}^{\infty} \frac{1}{n^2}$).

2.2. **Summation, and a Riemann sum.** Consider the partial sum

$$\sum_{n=1}^{N}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right)\,.$$

It includes the reciprocals of all odd numbers between 1 and $4N-1$, but only the reciprocals of the even numbers between 2 and $2N$. Adding and subtracting the "missing" terms (the reciprocals of the even intevers between $2N+2$ and $4N$) we get:

$$\begin{aligned}\sum_{n=1}^{N}\left(\frac{1}{4n-3}+\frac{1}{4n-1}-\frac{1}{2n}\right) &= \sum_{\substack{n=1\\ n \text{ odd}}}^{4N}\frac{1}{n} - \sum_{\substack{n=1\\ n \text{ even}}}^{2N}\frac{1}{n} \\ &= \sum_{\substack{n=1\\ n \text{ odd}}}^{4N}\frac{1}{n} - \sum_{\substack{n=1\\ n \text{ even}}}^{2N}\frac{1}{n} - \sum_{\substack{n=2N+1\\ n \text{ even}}}^{4N}\frac{1}{n} + \sum_{\substack{n=2N+1\\ n \text{ even}}}^{4N}\frac{1}{n} \\ &= \sum_{n=1}^{4N}\frac{(-1)^{n-1}}{n} + \sum_{\substack{n=2N+1\\ n \text{ even}}}^{4N}\frac{1}{n}\,.\end{aligned}$$

We now take the limit of each part of this sum separately. We recognize the first piece, $\sum_{n=1}^{4N} \frac{(-1)^{n-1}}{n}$, as a partial sum of the alternating harmonic series from Section 1, so

$$\lim_{N\to\infty} \sum_{n=1}^{4N} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \log 2\,.$$

For the second part, the even numbers between $2N+1$ and $4N$ are the numbers of the form $2k$ where $N < k \leq 2N$. Writing $k = N + i$ we see that

$$\sum_{\substack{n=2N+1 \\ n \text{ even}}}^{4N} \frac{1}{n} = \sum_{i=1}^{N} \frac{1}{2(N+i)} = \sum_{i=1}^{n} \frac{1}{2} \frac{1}{1+\frac{i}{N}} \cdot \frac{1}{N}\,.$$

We now recognize this expression as the (right-hand-rule) *Riemann sum* for the integral

$$\int_0^1 \frac{1}{2(1+x)}\,\mathrm{d}x = \frac{1}{2}\left[\log(1+x)\right]_{x=0}^{x=1} = \frac{1}{2}\left(\log 2 - \log 1\right) = \frac{1}{2}\log 2\,.$$

It follows that

$$\lim_{N\to\infty} \sum_{\substack{n=2N+1 \\ n \text{ even}}}^{4N} \frac{1}{n} = \frac{1}{2}\log 2$$

and hence that

$$\sum_{n=1}^{\infty} \left(\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{2n}\right) = \log 2 + \frac{1}{2}\log 2 = \frac{3}{2}\log 2\,.$$

3. Further exercizes

Problem 1. Find the sum of the rearrangement of the series where we take p positive terms followed by q negative ones (in the original series $p = q = 1$ and in the worked example $p = 2, q = 1$). Your answers will naturally depend on p and q.

Problem 2. For each real number α, find a rearrangement of the series converging to α.