Lecture 23, Solvable groups (1/12/2015)

Last time: nilpotence.

G nilpotent of degree $0 \Leftrightarrow G = \{1\}$

G nilpotent of deg $d+1 \Leftrightarrow G/Z(G)$ nilpotent of degree d

nilp. of deg $1 \Leftrightarrow G$ abelian, $\neq \{1\}$

nilp of deg $2 \Leftrightarrow G/Z(G)$ abelian $\neq \{1\}$

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def:

$Z^0(G) = \{1\}$, $Z^1(G) = Z(G)$, more generally $Z^i(G)$ normal, let $Z^{i+1}(G) \supseteq Z^i(G)$ to be the subgp such that

$$Z^{i+1}(G)/Z^i(G) = Z\left(G/Z^i(G)\right)$$

G nilpotent deg $d \Leftrightarrow Z^d(G) = G$, $Z^{d-1}(G) \neq G$.

Saw this is a normal series: $Z^i(G) \triangleleft Z^{i+1}(G)$

i.e. if G is nilpotent, get a series

$$\{1\} = Z^0(G) \triangleleft Z^1(G) \triangleleft \cdots \triangleleft Z^d(G) = G$$

with $Z^{i+1}(G)/Z^i(G)$ abelian.

Def: G is solvable if have a ~~series~~ normal series with abelian quotients

Today: solvability

Suppose $\{e\} = G_0 \lhd G_1 \lhd G_2 \lhd \cdots \lhd G_n = \{G\}$ is a normal series.

In a way G is "made up" from factors

$$G_1/G_0,\ G_2/G_1,\ \ldots,\ G_n/G_{n-1}.$$

$$\begin{tikzcd}[row sep=small] G_n \arrow[d, dash, dotted] \\ G_{i+1} \arrow[d, dash] \arrow[dr, dash] & \\ G_i \arrow[d, dash, dotted] & H \arrow[l, dash] \\ G_2 \arrow[d, dash] \\ G_1 \arrow[d, dash] \\ G_0 \end{tikzcd}$$

suppose G_{i+1}/G_i not simple - has normal subgp N.

$$\{e\} \subsetneq N \subsetneq G_{i+1}/G_i$$

Then N corresponds to a subgp $G_i < H < G_{i+1}$

$$(N = H/G_i < G_{i+1}/G_i)$$

$N \lhd G_{i+1}/G_i \Rightarrow H \lhd G_{i+1}$ (correspondence thm)

Also, $G_i \lhd H$ (it's normal in G_{i+1})

So can insert H in the normal series, get a "refined" one.

Suppose G is <u>finite</u>. Then cannot refine forever:

$$\prod_{i=1}^{n} (\# G_i/G_{i-1}) = \# G. \quad \text{(Lagrange + multiplicativity in towers)}$$

Assuming wlog $G_{i+1} \neq G_i$, $[G_{i+1} : G] \geq 2$

so $n \leq \log_2 \# G$.

So after finitely many steps cannot refine $\Rightarrow G_{i+1}/G_i$ all simple.

<u>Thm:</u> (Jordan-Hölder) Suppose G has a finite normal series with simple quotients. Then the list of quotients (with multiplicities) depends only on G.

Example 6: $G = C_{pq}$ Then $\{1\} \lhd C_p \lhd C_{pq}$ is a normal series
$\{1\} \lhd C_q \lhd C_{pq}$ is also

Two kinds of finite simple groups: (1) C_p
(2) non-abelian

i.e. if G is finite have two possiblities for a normal series with simple quotients:

(1) all factors isom to C_p $\leftarrow$ G is solvable.
(2) some factor is a non-abelian simple group

Properties of solvability

Prop: Suppose G is solvable, $H < G$. Then H is solvable.

Pf: Say $\{1\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$ is a normal series G_i/G_{i-1} abelian.

let $H_i = H \cap G_i$. Then $G_i \subset G_{i+1} \Rightarrow H_i < H_{i+1}$

Next, $H_i \lhd H_{i+1}$: let $h \in H_{i+1}$. Then $h \in G_{i+1}$, so $h G_i h^{-1} = G_i$

Also, $hHh^{-1} = H$ so $h(G_i \cap H)h^{-1} = G_i \cap H$

What about H_{i+1}/H_i? let $x, y \in H_{i+1}$. Need to show x, y commute mod H_i.

equivalently, want to show $[x,y] \in H_i$.

First, $x, y \in H_{i+1} \subset H$ so $[x,y] \in H$.

Second, $x, y \in G_{i+1}$ and G_{i+1}/G_i is abelian, i.e. $[x,y] \in G_i$.

So indeed $[x,y] \in H \cap G_i = H_i$.

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different argument: let $q: G_{i+1} \to G_{i+1}/G_i$ be the quotient map.

consider restriction $f = q\restriction_{H_{i+1}}$, $f \in \mathrm{Hom}(H_{i+1}, G_{i+1}/G_i)$

$\mathrm{Ker}(f) = \mathrm{Dom}(f) \cap \mathrm{Ker}(q) = H \cap G_{i+1} \cap G_i = H \cap G_i = H_i$

By 1^{st} isom thm, f induces an isomorphism

$$H_i \backslash H_{i+1} \xrightarrow{f} \mathrm{Im}(f) \leq G_{i+1}/G_i.$$

i.e. $H_i \backslash H_{i+1}$ is isomorphic to a subgp of the commutative group $G_i \backslash G_{i+1}$.

Prop: If G is solvable, $N \trianglelefteq G$ then G/N is solvable

Pf. Let $f \in \mathrm{Hom}(G, H)$ want to prove $\mathrm{Im}(f)$ is solvable.

Let $G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_n = G$ be a normal series with $G_i \backslash G_{i+1}$ abelian.

Let $H_i = f(G_i)$. Then clearly $H_i \subset H_{i+1}$, $H_n = \mathrm{Im}(f)$

Let ~~$g \in G$~~. ~~then~~ $h \in H_{i+1}$, ~~then~~ say $h = f(g)$, $g \in G_{i+1}$

Then $h H_i h^{-1} = f(g) f(G_i) f(g)^{-1} = f(g G_i g^{-1}) \underset{G_i \trianglelefteq G_{i+1}}{=} f(G_i) = H_i$

Moreover, look at $f_{i+1} : G_{i+1} \to H_i \backslash H_{i+1}$

composition of $f|_{G_{i+1}}$ and quotient map: $G_{i+1} \xrightarrow{f} H_{i+1} \xrightarrow{q_{i+1}} H_{i+1}/H_i$.

$\mathrm{Ker}(f_{i+1}) = \{g \in G_{i+1} \mid q_{i+1}(f(g)) = e\} = \{g \in G_{i+1} \mid f(g) \in H_i\} \supseteq G_i$

$= \cancel{\{g \in G_{i+1} \mid \dots\} \supseteq G_i}$. $\quad \overset{\shortparallel}{G_i N \cap G_{i+1}}$

f_{i+1} is surjective ($f|_{G_{i+1}}(G_{i+1}) = H_{i+1}$, q_{i+1} is surj)

so $\mathrm{Ker}(f)_{i+1} \backslash G_{i+1} \cong H_{i+1}/H_i$. By 3^{rd} isom thm, $G_{i+1}/\mathrm{Ker}(f) \cong (G_{i+1}/G_i) \big/ (\mathrm{Ker}(f)/G_i)$.

so H_{i+1}/H_i is isom to a quotient of an abelian gp, hence abelian

(by Hand in G/N : $H_i = G_i N / N \leq G/N$)

Thm: Suppose $N \lhd G$, both N, G/N are solvable. Then ~~B~~ G is.

$\{e\} \lhd N \lhd G$

Pf: Say $\{e\} = N_0 \lhd N_1 \cdots \lhd N_r = N$ is a normal series in N.

Say $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_s = G/N$ " " " " " G/N

Let G_{r+j} be the subgp of G containing N corresponding to H_j;

let $G_i = N_i$ if $i \le r$.

Then $\{e\} = G_0 \lhd G_1 \cdots \lhd G_r = N \lhd G_{r+1} \lhd \cdots \lhd G_{r+s} = G$

Because correspondence preserves normality: $H_j \lhd H_{j+1}$
$$\Downarrow$$
$$G_{r+j} \lhd G_{r+j+1}$$

& abelian quotients: corresp thm preserves quotients

Example: $B_2 = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix} \middle| \begin{array}{l} a, d \in F^\times \\ b \in F \end{array} \right\} \subset GL_2(F)$

contains $U_2 = \left\{ \begin{pmatrix} 1 & b \\ & 1 \end{pmatrix} \right\}$ normal; let $f: B_2 \to (F^\times)^2$ be

$f\begin{pmatrix} a & b \\ & d \end{pmatrix} \to (a,d)$.

Then $U_2 = \operatorname{Ker}(f)$

$\operatorname{Ker}(f)$ is solvable. (U_2 is nilpotent, abelian)

$\operatorname{Im}(f)$ is commutative ($\cong (F^\times)^2$)

By thm, B_2 is solvable.

Ex: B_2 not nilpotent. $Z(B_2/Z(B_2)) = \{e\}$

Recap: (1) Solvability $\leftrightarrow$ normal series + abelian quotients
(2) G solvable iff $N, G/N$ are ($N \trianglelefteq G$)
(3) played with subgps, quotients, isom thms

Thm (P. Hall, 1928) Let G be a finite solvable gp of order mn, $\gcd(m,n)=1$
Then (1) G has subgps of order m
(2) They are all conjugate.

(Sylow: case m = prime power)
(note: A_5 has order $60 = 4 \cdot 15$ but no subgp of order 15, index 4)