lecture 22, 26/11/2015 : Solvable & nilpotent groups

Two from PS10:

1) (c): A abelian, A/A_{tors} is torsion-free

Pf: $q: A \to A/A_{tors}$ is the quotient map

suppose $\bar{a} \in (A/A_{tors})_{tors}$, say $\bar{a} = q(a)$

By assumption for some $k \neq 0$, $\bar{a}^k = e$, i.e $q(a)^k = e$, i.e

$q(a^k) = e$, so $a^k \in \ker(q)$, i.e $a^k \in A_{tors}$

This means $\exists l$ s.t. $(a^k)^l = e$, so $a^{kl} = e$, $a \in A_{tors}$,

and $\bar{a} = q(a) = e$.

Observe: we showed: if G any gp, $N \trianglelefteq G$, $N \subset G_{tors}$

and if $\underbrace{g \in G \text{ is torsion mod } N}_{gN \in (G/N)_{tors}}$, then $g \in G_{tors}$.

4) (b): Say $G/Z(G)$ abelian. Show G_{tors} is a subgp

Pf: let $x, y \in G_{tors}$. Need to show $xy \in G_{tors}$

Saw: $[x,y] \in Z(G)_{tors}$. if $[x,y] = e$ then $xy \in G_{tors}$

consider images $\bar{x}, \bar{y}$ of x, y in $G/Z(G)_{tors}$.

$Z(G)_{tors}$ is a subgp of $Z(G)$ ($Z(G)$ is abelian)

is normal in G becase for $g \in G$, $z \in Z(G)_{tors} \subset Z(G)$,

$gzg^{-1} = z$.

$\bar{x}, \bar{y}$ torsion in $G/Z(G)_{tors}$ (in general: if $x^k = e$ then $f(x)^k = e$ for any hom $f: G \to H$)

Also, $\bar{x}, \bar{y}$ commute:

$$[\bar{x}, \bar{y}] = [q(x), q(y)] = q(x)q(y)q(x)^{-1}q(y^{-1}) = q(xyx^{-1}y^{-1}) = q([x,y]) = e$$

where $q: G \to G/Z(G)_{tors}$ is quot. map

so $\bar{x}\bar{y} = q(xy)$ is torsion (if $x^k = e$, $y^\ell = e$, $(\bar{x}\bar{y})^{k\ell} = e$)
By observation above, $xy \in G_{tors}$ too.

Suppose $G/Z(G)$ is two ~~step~~-nilpotent
(say "G is three-step nilpotent"). Again G_{tors} is a subgp

Let $x, y \in G_{tors}$. Consider images of x, y in $G/Z(G)$. These are torsion elements there, by $(G/Z(G))_{tors}$ is a subgp, so $(xy)\cdot Z(G)$ is torsion, i.e. $(xy)^k \in Z(G)$ for some k.
Not done. don't know $(xy)^k \in Z(G)_{tors}$.

<u>Def</u>: G is 0-step nilpotent if $G = \{e\}$
G is $(k+1)$-step " if G is not k-step nilpotent, but $G/Z(G)$ is.

<u>Example</u>: finite p-groups are nilpotent.

Pf: By induction on order: If G finite p-gp, $Z(G) \neq \{1\}$, so $G/Z(G)$ is smaller, by induction nilpotent.

show: $\#G = p^k$, G nilp of order $\le k$

<u>Fact</u>: finite gp G is nilpotent iff $G \cong \prod_p P_p$ (pdt of Sylow subgps)
Pf: Study $Z(G)$, $G/Z(G)$, put together

Suppose G is k-step nilpotent. Let $\gamma_0(G) = \{e\} \subset \gamma_1(G) = Z(G)$
define $\gamma_2(G)$ to be the subgp s.t. $\gamma_2(G)/Z(G) = Z(G/Z(G))$

(Correspondence: $\left\{ \begin{array}{c} \text{subgps of } G \\ \text{containing } Z(G) \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{subgps of} \\ G/Z(G) \end{array} \right\}$)

note: $\gamma_2(G) = \{ z \in G \mid z \text{ central "mod centre"} \} = \{ z \in G \mid \forall g \in G: [z,g] \in Z(G) \}$

def $\gamma_{i+1}(G) \supseteq \gamma_i(G)$ by $\gamma_{i+1}(G)/\gamma_i(G) = Z(G/\gamma_i(G))$

$Z\left(G/\gamma_i(G)\right) \triangleleft G/\gamma_i(G)$ so corresponding subgp $\gamma_{i+1}(G)$ is normal.

Different view of nilpotence: For any G, can define $\gamma_0 \subset \gamma_1 \subset \gamma_2 \subset \cdots$ "lower central series".

G is nilpotent if $\gamma_k(G) = G$ for some k.

Notes $\{\gamma_{i+1}\}$ normal in G, $\gamma_{i+1}(G)/\gamma_i(G)$ commutative

~~General Notation:~~ ~~Example:~~ Reminder (linear algebra) $T \in End(V)$ is nilpotent when $T^k = 0$ for some k.

Example: $U_n = \left\{ g \in GL_n(\mathbb{R}) \;\middle|\; \begin{array}{l} g \text{ upper-triangular} \\ diag(g) = (1,1,\dots,1) \end{array} \right\}$ if $g \in U_n$, $(g - I)$ and $\log(g)$ are nilpotent.

$U_2 = \left\{ \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \right\}$, $U_3 = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}$, ...

$Z(U_3) = \left\{ \begin{pmatrix} 1 & 0 & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\}$ $\quad U_3/Z(U_3) = \left\{ \begin{pmatrix} 1 & x & \\ & 1 & y \\ & & 1 \end{pmatrix} \right\} \cong \mathbb{F}^2$.

$$\begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ & 1 & y' \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & x+x' & * \\ & 1 & y+y' \\ & & 1 \end{pmatrix}$$

$Z(U_4) = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}$ $\quad \gamma_2(U_4) = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ & 1 & 0 & * \\ & & 1 & 0 \\ & & & 1 \end{pmatrix} \right\}$, $\gamma_3(U_4) = U_4$

For $g \in U_n$, $\log(g) = \log(I + (g - I)) \overset{\subseteq \mathfrak{u}_3}{=} \sum_{i=1}^{n} \frac{(-1)^{i+1}}{i} \cdot (g - I)^i$

Def: G is a gp. Call a chain of subgps

$$\{e\} = G_0 \subset G_1 \subset \cdots \subset G_k = G$$

a <u>normal series</u> if $G_i \trianglelefteq G_{i+1}$ for each i.

(don't need $G_i \trianglelefteq G$).

Always have $\{e\} \trianglelefteq G$

Idea: Understand G from quotients G_{i+1}/G_i.

Def: Say G is <u>solvable</u> if G has a normal series with G_{i+1}/G_i abelian for all i.

Example: nilpotent gps. Also $B_n = \{ g \in GL_n \mid g \text{ upper triangular} \}$

Say G is solvable of deg d if has normal series with d terms & abelian quotients.

Thm: (Galois) let $f \in \mathbb{Q}[x]$ be a polynomial, let $\Sigma \subset \mathbb{C}$ be the field generated by roots of f. ("splitting field of f")

let $Gal(f) = Aut(\Sigma) = \left\{ \varphi : \Sigma \to \Sigma \;\middle|\; \begin{array}{l} \varphi \text{ bijective} \\ \varphi(x+y) = \varphi(x) + \varphi(y) \\ \varphi(xy) = \varphi(x)\varphi(y) \end{array} \right\}$

Then (1) $\# Gal(f)$ finite.

(2) roots of f can be expressed using radicals iff $Gal(f)$ is solvable.

Example: $f(x) = ax^2 + bx + c$, roots $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

(if $\sqrt{b^2 - 4ac} \in \mathbb{Q}$, $Gal(f) = \{1\}$

if $\sqrt{} \notin \mathbb{Q}$, $Gal(f) \cong C_2$)

Abel: $Gal(f)$ commutative $\Rightarrow f$ is solvable by radicals.