lecture 19, 17/11/2015

More sylow theorems

Previously: Thm: G finite, order $n = p^k \cdot m$, $p \nmid m$, p prime.
Then $\mathrm{Syl}_p(G) = \{P < G \mid \#P = p^k\}$ is non-empty,
a conjugacy class, has $n_p(G) \mid m$ elements, where $n_p(G) \equiv 1 \ (p)$.
(also every p-subgp $\subseteq$ p-Sylow subgp)

Application: $\#G = 12$, G is one of $C_{12}, C_2 \times C_6, A_4, C_2 \times S_3, C_4 \ltimes C_3$.

ideas: ① numerology (divisibility & congruence)

② if $n_p(G) = 1$ ideas of semidirect pdt
automorphisms

③ if G acts on X get hom $G \to S_X$.

Example: no simple group of order 30.

Pf: Let G be such a gp. Note: $30 = 2 \cdot 3 \cdot 5$.

$n_2(G) \in \{1, 3, 5, 15\}$, $n_3(G) \in \{1, \cancel{2}, \cancel{5}, 10\}$, $n_5(G) \in \{1, \cancel{2}, \cancel{3}, 6\}$

If is simple, $n_3(G), n_5(G) \neq 1$ (else P_3 or P_5 would be normal)

so $n_3(G) = 10$, $n_5(G) = 6$.

④ count elements: 10 3-sylow subgps, each having 2 elements of order 3.
all disjoint (cyclic ~~gp~~ gps of order p have no non-trivial proper subgps)

so 20 elements of order 3.

Similarly $6 \cdot 4 = 24$ elements of order 5.

But can't have 44 elements in G, so either $n_3(G) = 1$ or $n_5(G) = 1$.

More: let P_3, P_5 be a 3- & 5-sylow subgps. Then $P_3 P_5$ is a subgp (a semidirect pdt) of order 15. By classification of groups of order pq ($\nexists$ since $5 \not\equiv 1\ (3)$), $H = P_3 P_5$ is a direct pdt, $\simeq C_{15}$

$C_3 \times C_5$

So G has a subgp $H \simeq C_{15}$, of index 2 hence normal.

Thus $G = P_2 H$ as a semidirect pdt.

$$G \simeq C_2 \ltimes (C_3 \times C_5)$$

classified by aut of order 2 of $C_3 \times C_5$ by which the C_2 acts

$$\mathrm{Aut}(C_3 \times C_5) \simeq \mathrm{Aut}(C_{15}) \simeq (\mathbb{Z}/15\mathbb{Z})^\times \underset{\text{CRT}}{\simeq} (\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times$$

$$\simeq C_2 \times C_4.$$

Now C_n has at most 1 subgp hence 1 element of order 2,

so elements of order 1, 2 in $(\mathbb{Z}/3\mathbb{Z})^\times \times (\mathbb{Z}/5\mathbb{Z})^\times$ are $\{(\pm 1, \pm 1)\}$

<u>Conclusion</u>: Gps of order 30 are:

1. $C_2 \times C_{15} \simeq C_{30}$ $\longleftrightarrow$ $(+1, +1) \leftrightarrow [1]_{15} \in (\mathbb{Z}/15\mathbb{Z})^\times$
2. $C_3 \times (C_2 \ltimes C_5) \simeq C_3 \times D_{10}$ $\longleftrightarrow$ $(+1, -1) \leftrightarrow [4]_{15}$
3. $C_5 \times D_6 \simeq C_5 \times S_3$ $\longleftrightarrow$ $(-1, 1) \leftrightarrow [11]_{15}$
4. $C_2 \ltimes C_{15} \simeq D_{30}$ $\longleftrightarrow$ $(-1, -1) \leftrightarrow [14]_{15}$

<u>Remark</u>: H normal, contains a P_3, a P_5 so contains $\mathrm{Syl}_3(G)$, $\mathrm{Syl}_5(G)$

But H commutative, $n_3(H) = n_5(H) = 1$, so $n_3(G) = n_5(G) = 1$

<u>Example</u>: Let G be a simple group of order $60 = 2^2 \cdot 3 \cdot 5$.

Then $G \simeq A_5$.

numerology: $n_2(G) \in \{\cancel{1}, 3, 5, 15\}$, $n_3(G) \in \{\cancel{1}, 4, 10\}$

$n_5(G) \in \{\cancel{1}, 6\}$

G simple $\Rightarrow$ every hom $f: G \to H$ is either <u>trivial</u> ($f(g) = e$)

or <u>injective</u> (because $\mathrm{Ker}(f)$ is one of $G, \{e\}$)

This excludes $n_2(G)=3$, or $n_3(G)=4$: the resulting hom to S_3 or S_4
won't be injective ($\#G = 60 > 24, 6$)
won't be trivial (G acts transitively on p-Sylow subgps, so action is non-trivial)

<u>Conclusion;</u> $n_3(G)=10$, $n_5(G)=6$, $n_2(G) \in \{5, 15\}$.

<u>Case I:</u> $n_2(G)=5$. Then conjugation action on $Syl_2(G)$ gives hom $f: G \to S_5$. This is non-trivial, hence injective,
image is a subgp of S_5 of order 60, index 2, hence normal
But normal subgps of S_5 are $\{1\}, A_5, S_5$, so $f(G)=A_5$ and $G \cong A_5$

<u>Case II:</u> $n_2(G)=15$. Recall $n_3(G)=10$, $n_5(G)=6$,
so G has 20 elements of order 3, 24 elements of order 5
so at most $60-20-24-1=15$ elements of order 2 or 4.
Can't have all P_2's disjoint (then $15 \cdot 3$ elements of order 2 or 4)
let $x \in G$ be a non-identity element belonging to two 2-sylow subgps
The 2-sylow subgps of G have order $4=2^2$ so are commutative,
So $Z_G(x)$ contains both 2-sylow subgps containing x, so it properly contains them (they are distinct, of order 4)
So index of $Z_G(x)$ properly divides $[G:P_2]=15$.
So size of conjugacy class of x properly divides 15.
The class doesn't have size 1 ($Z(G) \neq G$ (not commutative) so $Z(G)=\{1\}$)
doesn't have size 3 (G doesn't act transitively on sets of size 3)
So $[G:Z_G(x)]=5$ and conjugacy class of x furnishes a non-trivial G-set of size 5.
(we found if P_2, P_2' are non-disjoint 2-sylow subgps, $\langle P_2, P_2' \rangle$ has index 5, order 12