Lecture 12: Conjugacy 22/10/2015

(1) Hom point of view

(2) Conjugacy

(3) ?

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(1) ~~Re~~ Fix gp $G$, set $X$. <u>Recall</u>: <u>Action</u> of $G$ on $X$ is a map $\cdot : G \times X \to X$ s.t. $\begin{cases} g\cdot(h\cdot x) = (gh)\cdot x \\ e\cdot x = x \end{cases}$ $x \in X$, $g,h \in G$

<u>Saw</u>: For "regular" action of $G$ on itself by left translation:

$g\cdot x = gx$

mult in $G$

Defining $\sigma_g(x) = g\cdot x = gx$ gave a hom $G \to S_G$, (Cayley: an injection, $G \simeq \text{Image}(\sigma) < S_G$)

<u>Lemma</u>: (1) Let $G$ act on $X$. Then $\sigma_g(x) = g\cdot x$ defines an element $\sigma_g \in S_X$.

(2) $g \mapsto \sigma_g$ is a hom $G \to S_X$.

(3) This association is a bijection $\left\{\begin{matrix}\text{actions} \\ \text{of } G \\ \text{on } X\end{matrix}\right\} \longrightarrow \text{Hom}(G, S_X)$

<u>Pf</u>: (1) + (2): $\sigma_g$ is a function $X \to X$

Axioms of group action $\iff$ $\sigma_e = \text{id}_X$, $\sigma_g \circ \sigma_h = \sigma_{gh}$.

Now $\sigma_g \circ \sigma_{g^{-1}} = \sigma_{gg^{-1}} = \sigma_e = \text{id}_X$, $\sigma_{g^{-1}} \circ \sigma_g = \sigma_{g^{-1}g} = \sigma_e = \text{id}_X$ so $\sigma_g, \sigma_{g^{-1}} : X \to X$ are inverses so both are bijections, $\sigma_g \in S_X$ and $\sigma_\cdot \in \text{Hom}(G, S_X)$

(if $\sigma_\cdot \in \text{Hom}(G, S_X)$ then $(\sigma_g)^{-1} = \sigma_{g^{-1}}$ which helps leads to proof)

(3) Suppose $\cdot, *$ are actions of $G$ on $X$.

let $\begin{cases} \sigma_g(x) = g\cdot x \\ \tau_g(x) = g * x \end{cases}$ Suppose $\sigma_g = \tau_g$ for all $g$ ($\sigma_\cdot = \tau_\cdot$ as homs)

Then for any $x \in X$, $g \in G$: $g\cdot x = \sigma_g(x) = \tau_g(x) = g * x$ so $\cdot = *$

Also, if $\sigma \in \mathrm{Hom}(G, S_X)$ define action of $G$ on $X$ by $g\cdot x = \sigma_g(x)$. Then $e\cdot x = \sigma_e(x) \underset{\sigma \text{ is a hom}}{=} e_{S_X}(x) \underset{e_{S_X} = \mathrm{id}\text{ perm}}{=} \mathrm{id}_X(x) = x$

& $g\cdot(h\cdot x) = \sigma_g(\sigma_h(x)) \underset{\text{def } \circ}{=} (\sigma_g \circ \sigma_h)(x) \underset{\sigma \text{ is a hom}}{=} \sigma_{gh}(x) \underset{\text{def of } \cdot}{=} (gh)\cdot x$

[Ex: if $G$ acts on $X$, $f \in \mathrm{Hom}(H, G)$ get "pullback action" of $H$ on $X$ by $h\cdot x \overset{\text{def}}{=} f(h)\cdot x$.]

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② Conjugation (fix a gp $G$)

Def: For $g \in G$, $x \in G$ write ${}^g x = gxg^{-1}$, also $\gamma_g(x) = gxg^{-1}$!

Lemma: This defines an action of $G$ on itself, by group automorphisms.
(i.e. $\gamma_\cdot \in \mathrm{Hom}(G, \mathrm{Aut}(G))$, $\mathrm{Aut}(G) = \mathrm{Hom}(G,G) \cap S_G$)

Pf: check. (See PS2 for the case $G = S_X$)

Def: Say "$x$ is conjugate to $y$" if ${}^g x = y$ for some $g \in G$.

Example 1: $G = GL_n(F)$ conjugate = "similar".

Thm: let $g \in GL_n(\mathbb{R})$ be symmetric ($g = g^t$). Then $g$ is conjugate to a diagonal matrix. (by an orthogonal one)

Example 2: In $S_n$, $\sigma$ is conj. to $\tau$ iff they have same cycle structure

Lemma: Conjugacy is an equivalence relation.

Pf: PS3 problem 2(a).

Def: The equivalence classes are called "conjugacy classes".

Example: The class of $e$ is $\{e\}$ : $geg^{-1} = gg^{-1} = e$.

Example: The class of $g$ is $\{g\}$ iff $g \in Z(G)$

($g$ is central iff all $h$, $hgh^{-1} = g \Leftrightarrow hg = gh$)

Why care about this? (1) This is an action by automorphisms

(If $x, y$ are conjugate, they have same group-theory properties)

(2) These are readily available.

($G$ abelian $\Leftrightarrow$ every conjugacy class is a singleton)

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Remark: $\gamma_{\cdot} \in \mathrm{Hom}(G, \mathrm{Aut}(G))$ so its image is a subgp.

Called "inner automorphism", denoted $\mathrm{Inn}(G)$

Fact: $\mathrm{Inn}(G) \triangleleft \mathrm{Aut}(G)$ ($f \in \mathrm{Aut}$, then $f \circ \gamma_g \circ f^{-1} = \gamma_{f(g)}$)

Ker of $\gamma$ is $Z(G)$, by isom thm $G/Z(G) \cong \mathrm{Inn}(G)$

Def: $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ called "outer aut. gp".

Example: $\mathrm{Aut}(\mathbb{Z}^d) = GL_d(\mathbb{Z})$ all outer ($\mathrm{Inn}(\mathbb{Z}^d) = \{id\}$)

Example: $\mathrm{Out}(S_n) = \{1\}$ except $\mathrm{Out}(S_6) \cong C_2$.

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Lemma: There is a bijection between the conjugacy class of $x$ and the coset space $G/Z_G(x)$. (in particular, the class has $[G : Z_G(x)]$ elements)

Pf: Map $gZ_G(x) \mapsto {}^g x$

checks (1) well-def
(2) bijection

(1) say $g = g'z$, $z \in Z_G(x)$ then

$${}^g x = {}^{g'z} x = (g'z)x(g'z)^{-1} = g'zxz^{-1}(g')^{-1}$$
$$\underset{\substack{\uparrow \\ z \in Z_G(x)}}{=} g' x (g')^{-1} = {}^{g'} x$$

(2) surjectivity: if $y = {}^g x$
then $gZ_G(x)$ maps to $y$

injectivity: if $gZ_G(x)$, $hZ_G(x)$ map to $y$ have

$$gxg^{-1} = y = hxh^{-1} \quad \text{so} \quad h^{-1}gxg^{-1}h = x$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \overset{\shortparallel}{(h^{-1}g)x(h^{-1}g)^{-1}}$$

so $h^{-1}g \in Z_G(x)$ so $g = hz$ for some $z \in Z_G(x)$
and $gZ_G(x) = hZ_G(x)$.