**Math 101 – SOLUTIONS TO WORKSHEET 33**
**TAYLOR SERIES AND DERIVATIVES**

The *Taylor series* of $f(x)$ *centered at* $a$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

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(1) Find the MacLaurin series of $f(x) = e^x$.
**Solution:** For each $n$ we have $f^{(n)}(x) = e^x$ so $f^{(n)}(0) = e^0 = 1$. The series is therefore

$$\boxed{\sum_{n=0}^{\infty} \frac{1}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}\,.}$$

(2) (Final 2014) Find the Taylor series $g(x) = \log x$ centered at $a = 2$, as well as its radius of convergence.
**Solution:** $g'(x) = \frac{1}{x}$, $g''(x) = -\frac{1}{x^2}$, $g^{(3)}(x) = \frac{1\cdot 2}{x^3}$, $g^{(4)}(x) = -\frac{1\cdot 2\cdot 3}{x^4}$, and in general $g^{(n)}(x) = (-1)^{n-1}\frac{(n-1)!}{x^n}$. So for $n \geq 1$ we have $g^{(n)}(2) = (-1)^{n-1}\frac{(n-1)!}{2^k}$ and the Taylor series is

$$\begin{aligned}\log 2 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!}{2^n n!}(x-2)^n &= \log 2 + \sum_{n=1}^{\infty}(-1)^n\frac{(n-1)!}{2^n n(n-1)!}(x-2)^n \\ &= \log 2 + \sum_{n=1}^{\infty}(-1)^n\frac{(x-2)^n}{2^n n}\,.\end{aligned}$$

For the radius of convergence we compute $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{2^{n+1}(n+1)}\big/\frac{(-1)^n}{2^n n}\right| = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{2^n}{2^{n+1}} = \frac{1}{2}\lim_{n\to\infty}\frac{1}{1+\frac{1}{n}} = \frac{1}{2}$ so we have $R = 2$.
**Solution:** We have

$$\log x = \log(2+(x-2)) = \log\left(2\left(1+\frac{x-2}{2}\right)\right) = \log 2 + \log\left(1+\frac{x-2}{2}\right).$$

We know that $\log(1+u) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}u^n$ and it follows that

$$\log x = \log 2 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(\frac{x-2}{2}\right)^n.$$

The logarithm series converges for $-1 < u \leq 1$ so our series will converge for $-1 < \frac{x-2}{2} \leq 1$ that is $-2 < x-2 \leq 2$ so the radius of convergence is 2.

(3) (Final 2014) Let $\sum_{n=0}^{\infty} c_n x^n$ be the MacLaurin series for $e^{3x}$. Find $c_5$.
**Solution:** Knowing that $e^u = \sum_{n=0}^{\infty}\frac{u^n}{n!}$ we have $e^{3x} = \sum_{n=0}^{\infty}\frac{3^n}{n!}x^n$ so $c_5 = \frac{3^5}{5!}$.

(4) (Final 2013) Let $f(x) = x^2\sin(x^3)$. Find $f^{11}(0)$.
**Solution:** We know that $\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$ so

$$x^2\sin(x^3) = x^2\left(x^3 - \frac{x^9}{3!} + \cdots\right) = x^5 - \frac{x^{11}}{3!} + \cdots.$$

It follows that $\frac{f^{(11)}(0)}{11!} = \frac{1}{3!}$ so $f^{(11)}(0) = \frac{11!}{3!}$.

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