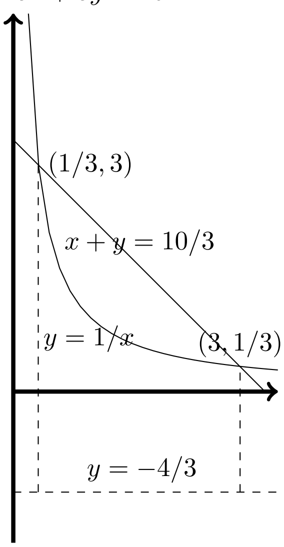
Math 101 – SOLUTIONS TO WORKSHEET 9
SOLIDS OF REVOLUTION,

(1) Solids of revolution

(a) (Final 2014, variant) Find the volume of the solid generated by rotating the finite region bounded by $y = \frac{1}{x}$ and $3x + 3y = 10$ about the line $y = -\frac{4}{3}$. It will be useful to sketch the region first.



Solution: The intersection points are where $x + \frac{1}{x} = \frac{10}{3}$ that is where $x^2 - \frac{10}{3}x + 1 = 0$ that is where $x = \frac{10/3 \pm \sqrt{\frac{100}{9} - 4}}{2} = \frac{10 \pm \sqrt{64}}{6} = \frac{5 \pm 4}{3} = \frac{1}{3}, 3$. Setting $f(x) = \frac{10}{3} - x$ and $g(x) = \frac{1}{x}$ the region is $\left\{(x, y) \mid \frac{1}{3} \le x \le 3,\ g(x) \le y \le f(x)\right\}$; the cross-sections when revolving about the line $x = -\frac{4}{3}$ are annuli with inner radius $g(x) + \frac{4}{3}$, outer radius $f(x) + \frac{4}{3}$ and therefore area $\pi\left(\left(f(x) + \frac{4}{3}\right)^2 - \left(g(x) + \frac{4}{3}\right)^2\right)$ so the volume is:

$$\begin{aligned}
\pi \int_{x=1/3}^{x=1} \left(\left(\frac{10}{3} - x + \frac{4}{3}\right)^2 - \left(\frac{1}{x} + \frac{4}{3}\right)^2\right) \mathrm{d}x &= \pi \int_{x=1/3}^{x=3} \left(\frac{196}{9} - \frac{28}{3}x + x^2 - \frac{1}{x^2} - \frac{8}{3x} - \frac{16}{9}\right) \mathrm{d}x \\
&= \pi \int_{x=1/3}^{x=3} \left(20 - \frac{28}{3}x + x^2 - \frac{1}{x^2} - \frac{8}{3x}\right) \mathrm{d}x \\
&= \pi \left[20x - \frac{14}{3}x^2 + \frac{x^3}{3} + \frac{1}{x} - \frac{8}{3}\log|x|\right]_{x=1/3}^{x=3} \\
&= \pi \left[\left(60 - 42 + 9 + \frac{1}{3} - \frac{8}{3}\log 3\right) - \left(\frac{20}{3} - \frac{14}{27} + \frac{1}{81} + 3 - \frac{8}{3}\log\frac{1}{3}\right)\right] \\
&= \pi \left[18\frac{14}{81} - \frac{16}{3}\log 3\right] = 18\frac{14}{81}\pi - \frac{16\pi}{3}\log 3\,.
\end{aligned}$$

(b) The area between the y-axis, the curve $y = x^2$ and the line $y = 4$ is rotated about the y-axis. What is the volume of the resulting region?

Solution: Slicing perpendicular to the y-axis, we need to evaluate

$$\int_{y=0}^{y=4} \pi x^2 \,\mathrm{d}y = \int_{y=0}^{y=4} \pi y \,\mathrm{d}y = \frac{\pi}{2}\left[y^2\right]_{y=0}^{y=4} = 8\pi\,.$$

Date: 20/1/2016, Worksheet by Lior Silberman.

(2) (Work)

(a) (Final, 2012) A tank in the shape of a hemispherical bowl of radius $R = 3$m is full of water. It is to be emptied through an outlet extending $H = 2$m above its top. Using the values $\rho = 1000\text{kg/m}^3$ for the density of water and $g = 9.8\text{m/s}^2$ for the acceleration due to gravity, find the work (in Joules) required to empty the tank completely. There is no need to simplify your answer but you must evaluate all integrals.

Solution: Slice the water into horizontal slabs perpendicular to the z-axis. Each slab has circular cross-section. The slab at height z above the bottom of the hemisphere is at distance $R - z$ below the centre of the sphere, so the radius of the slab satisfies

$$r^2 + (R - z)^2 = R^2$$

so

$$r^2 = 2Rz - z^2 \,.$$

The volume of the slab is therefore about $\pi r^2 \Delta z = \pi(2Rz - z^2)\Delta z$; the mass of the water in it is therefore

$$\Delta m \approx \pi\rho \left(2Rz - z^2\right) \Delta z \,.$$

This mass of water will be moved from the height z to the height $R + H$ against gravity, the required work is therefore

$$\begin{aligned} \Delta W &= \Delta m \cdot g \cdot (R + H - z) \\ &= \pi\rho g \left(2Rz - z^2\right)(R + H - z)\,\Delta z \,. \end{aligned}$$

The total work is therefore

$$\begin{aligned} \pi\rho g \int_{z=0}^{z=R} \left(2Rz - z^2\right)(R + H - z)\,\mathrm{d}z &= \pi\rho g \int_{z=0}^{z=R} \left(2Rz(R+H) - z^2(R+H) - 2Rz^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g \int_{z=0}^{z=R} \left(2R(R+H)z - (3R+H)z^2 + z^3\right)\mathrm{d}z \\ &= \pi\rho g \left[R(R+H)z^2 - \frac{3R+H}{3}z^3 + \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g \left((R+H)R^3 - (R + \frac{H}{3})R^3 + \frac{R^4}{4}\right) \\ &= \pi\rho g R^3 \left(\frac{R}{4} + \frac{2}{3}H\right) \\ &= 9,800\pi \cdot 27 \left(\frac{3}{4} + \frac{4}{3}\right) \mathrm{J}\,. \end{aligned}$$

Solution: Instead, let z measure depth below the surface of the water. Again we chop the water into horizontal slabs perpendicular to the z-axis. Then the slice at depth z has volume about $\pi r^2 \Delta z = \pi\left(R^2 - z^2\right)\Delta z$. This water will be moved distance $H + z$ against gravity, so the total work is

$$\begin{aligned} \int_{z=0}^{R} \pi\rho g \left(R^2 - z^2\right)(H + z)\,\mathrm{d}z &= \pi\rho g \int_{z=0}^{R} \left(R^2H + R^2z - z^2H - z^3\right)\mathrm{d}z \\ &= \pi\rho g \left[R^2Hz + \frac{R^2}{2}z^2 - \frac{H}{3}z^3 - \frac{z^4}{4}\right]_{z=0}^{z=R} \\ &= \pi\rho g \left[R^3H + \frac{R^4}{2} - \frac{H}{3}R^3 - \frac{R^4}{4}\right] \\ &= \pi\rho g R^3 \left(\frac{2}{3}H + \frac{R}{4}\right) \\ &= 9,800\pi \cdot 27 \left(\frac{4}{3} + \frac{3}{4}\right) \mathrm{J}\,. \end{aligned}$$