Math 101 – SOLUTIONS TO WORKSHEET 7
SUBSTITUTION, AREA BETWEEN CURVES

(1) Evaluate the integrals

(a) (Final, 2013) $\int_1^3 (2x-1)e^{x^2-x}\,\mathrm{d}x =$

Solution: Letting $u = x^2 - x$, $\mathrm{d}u = (2x-1)\,\mathrm{d}x$ we get $\int_{x=1}^{x=3}(2x-1)e^{x^2-x}\,\mathrm{d}x = \int_{u=0}^{u=6} e^u\,\mathrm{d}u = [e^u]_{u=0}^{u=6} = e^6 - 1$.

Solution: (Alternative) Using $u = x^2 - x$, $\mathrm{d}u = (2x-1)\,\mathrm{d}x$ we get $\int (2x-1)e^{x^2-x}\,\mathrm{d}x = \int e^u\,\mathrm{d}u = e^u + C = e^{x^2-x} + C$ so

$$\int_{x=1}^{x=3}(2x-1)e^{x^2-x}\,\mathrm{d}x = \left[e^{x^2-x}\right]_{x=1}^{x=3} = \boxed{e^6-1}.$$

Solution: (Alternative) Using $u = x^2 - x$, $\mathrm{d}u = (2x-1)\,\mathrm{d}x$ we get

$$\begin{aligned}\int_{x=1}^{x=3}(2x-1)e^{x^2-x}\,\mathrm{d}x &= \int_{x=1}^{x=3} e^u\,\mathrm{d}u = [e^u]_{x=1}^{x=3} \\ &= \left[e^{x^2-x}\right]_{x=1}^{x=3} = e^6 - 1 \\ &= [e^u]_{u=0}^{u=6} = e^6 - 1\,.\end{aligned}$$

Date: 18/1/2016, Worksheet by Lior Silberman. This instructional material is excluded from the terms of UBC Policy 81.

(b) (Final, 2012) $\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x =$

Solution: Write this as $\int_0^3 \sqrt{9-x^2}x\,\mathrm{d}x + \int_0^3 \sqrt{9-x^2}\,\mathrm{d}x$. The second term is the area of a quarter-circle of radius 3, so is $\frac{9}{4}\pi$. For the first term we use $u = 9-x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$ to see that

$$\begin{aligned}\int_{x=0}^{x=3} \sqrt{9-x^2}x\,\mathrm{d}x &= \int_{u=9}^{u=0} \sqrt{u}\left(-\frac{1}{2}\right)\mathrm{d}u = \frac{1}{2}\int_{u=0}^{u=9} u^{1/2}\,\mathrm{d}u \\ &= \frac{1}{2}\frac{2}{3}\left[u^{3/2}\right]_{u=0}^{u=9} = \frac{1}{3}9^{3/2} = 9\,.\end{aligned}$$

In conclusion, $\int_0^3 (x+1)\sqrt{9-x^2}\,\mathrm{d}x = \boxed{9+\frac{9}{4}\pi}$.

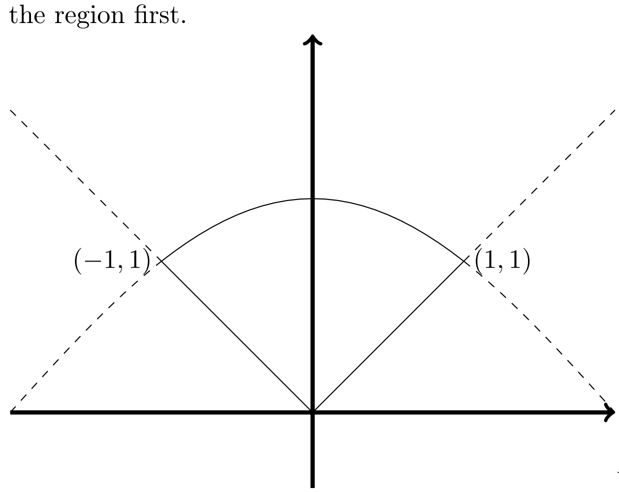
(2) Area between curves

(a) (Final, 2011) Find the total area of the finite place region lying between the curves $y=x$ and $y=x^3$.

Solution: The curves intersect where $x=x^3$, that is where $x(x+1)(x-1)=0$. On $[-1,0]$ we have $x \le x^3 \le 0$. On $[0,1]$ we have $0 \le x^3 \le x$. By symmetry the areas are equal, and the total area is therefore

$$\int_{-1}^{0}(x^3-x)\,\mathrm{d}x + \int_0^1 (x-x^3)\,\mathrm{d}x = 2\int_0^1 (x-x^3)\,\mathrm{d}x = 2\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{x=0}^{x=1} = \boxed{\frac{1}{2}}.$$

(b) (Final, 2014) Find the area of the finite region bounded between the two curves $y=\sqrt{2}\cos(x\pi/4)$ and $y=|x|$. It will be useful to sketch the region first.



Solution: We draw a sketch first. We conclude that the area is

$$\begin{aligned}2\int_0^1 \left(\sqrt{2}\cos(x\pi/4) - x\right)\mathrm{d}x &= 2\left[\sqrt{2}\frac{4}{\pi}\sin\frac{\pi x}{4} - \frac{x^2}{2}\right]_{x=0}^{x=1} \\ &= \frac{8\sqrt{2}}{\pi}\cdot\frac{1}{\sqrt{2}} - 2 = \boxed{\frac{8}{\pi}-2}.\end{aligned}$$