MATH 101: COMPUTING INTEGRALS FROM THE DEFINITION

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In this note I collect a few examples of computing integrals from the definition.

Problem 1. Compute $\int_0^1 e^x \,\mathrm{d}x$ using the definition. You may use the summation formula $\sum_{i=1}^n q^i = \frac{q^n - q}{q-1}$.

Solution: Dividing $[0,1]$ to n subintervals of length $\Delta x = \frac{1}{n}$, we get the points $x_i = \frac{i}{n}$ where $f(x_i) = e^{i/n}$ for $f(x) = e^x$. Thus using the right-side rule we calculate:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^n e^{i/n} \frac{1}{n} &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \left(e^{1/n}\right)^i \\
&= \lim_{n\to\infty} \frac{1}{n} \frac{\left(e^{1/n}\right)^n - \left(e^{1/n}\right)}{e^{1/n} - 1} \qquad (\text{formula with } q = e^{1/n}) \\
&= \lim_{n\to\infty} \frac{e - e^{1/n}}{n(e^{1/n} - 1)} \qquad \left((e^{1/n})^n = e\right) \\
&= \frac{\lim_{n\to\infty} \left(e - e^{1/n}\right)}{\lim_{n\to\infty} \frac{e^{1/n} - 1}{1/n}} .
\end{aligned}$$

as $n \to \infty$, $\frac{1}{n} \to 0$ so we need to compute

$$\frac{\lim_{x\to 0} (e - e^x)}{\lim_{x\to 0} \frac{e^x - 1}{x}} .$$

In the numerator there is no problem: $\lim_{x\to 0} (e - e^x) = e - e^0 = e - 1$. In the denominator, we recognize the limit as the definition of the derivative of $f(x) = e^x$ at $x = 0$. Since $f'(x) = e^x$ we have $\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1$ so

$$\frac{\lim_{x\to 0} (e - e^x)}{\lim_{x\to 0} \frac{e^x - 1}{x}} = \frac{e-1}{1} ,$$

and

$$\int_0^1 e^x \,\mathrm{d}x = e - 1 .$$

Problem 2. Compute $\int_a^b x \,\mathrm{d}x$ using the definition. You may use the summation formula $\sum_{i=1}^n i = \frac{n(n+1)}{2}$.

Solution: Dividing $[a,b]$ to n subintervals of length $\Delta x = \frac{b-a}{n}$, we get the points $x_i = a + \frac{b-a}{n} i$. We are working with $f(x) = x$ so $f(x_i) = x_i = \left(a + \frac{b-a}{n} i\right)$. Thus using the right-side rule we calculate:

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{b-a}{n} i\right) \frac{b-a}{n} &= \lim_{n\to\infty} \frac{b-a}{n} \left(\sum_{i=1}^{n} a + \sum_{i=1}^{n} \frac{b-a}{n} i \right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a \sum_{i=1}^{n} 1 + \frac{b-a}{n} \sum_{i=1}^{n} i \right) \\
&= \lim_{n\to\infty} \frac{b-a}{n} \left(a \cdot n + \frac{b-a}{n} \cdot \frac{n(n+1)}{2} \right) \\
&= \lim_{n\to\infty} \left(\frac{b-a}{n} \cdot (an) + \frac{b-a}{n} \cdot \frac{b-a}{n} \cdot \frac{n(n+1)}{2} \right) \\
&= \lim_{n\to\infty} \left((b-a)a + \frac{(b-a)^2}{2} \left(1 + \frac{1}{n}\right) \right) \qquad \left(\lim_{n\to\infty} \frac{1}{n} = 0 \right) \\
&= (b-a)a + \frac{(b-a)^2}{2} = (b-a)\left(a + \frac{b-a}{2}\right) \\
&= (b-a)\left(\frac{2a+b-a}{2}\right) = (b-a)\frac{b+a}{2} \\
&= \frac{b^2-a^2}{2}\,.
\end{aligned}$$

In other words:

$$\int_a^b x \, \mathrm{d}x = \frac{b^2-a^2}{2}\,.$$

Remark. If $0 < a < b$ the area under the graph is a trapeze and you can compute it geometrically. Of course, the integral can also be computed using the FTC and integration techniques. But if you are asked to use the definition then you can only get credit for a solution that computes the limit of the Riemann sums as above.

Problem 3 (Challenge)**.** Try computing $\int_a^b e^x \, \mathrm{d}x$ using the techniques above. (Hint: $e^{a+i\Delta x} = e^a \cdot \left(e^{\Delta x}\right)^i$).