MATH 100 – SOLUTIONS TO WORKSHEET 17
THE MVT

1. More minima and maxima

(1) Show that the function $f(x) = 3x^3 + 2x - 1 + \sin x$ has no local maxima or minima. You may use that $f'(x) = 9x^2 + 2 + \cos x$.

Solution: $f(x)$ is differentiable everywhere, so by Fermat's Theorem at any local extremum x_0 we'd have $f'(x_0) = 0$. However, at any x we have

$$f'(x) = 9x^2 + 2 + \cos x \geq 0 + 2 - 1 = 1 > 0\,.$$

(2) Let $g(x) = xe^{-x^2/8}$ so that $g'(x) = \left(1 - \frac{x^2}{4}\right) e^{-x^2/8}$, find the global minimum and maximum of g on

(a) $[-1, 4]$ (b) $[0, \infty)$

Solution: g is differentiable everywhere, so it's enough to consider its critical points and the endpoints of the intervals. Since $e^a \neq 0$ for all a, we have $g'(x) = 0$ iff $1 - \frac{x^2}{4} = 0$ i.e. iff $x = \pm 2$.

(a) On $[-1, 4]$ the only critical point is $x = 2$ and we have $f(-1) = -e^{-1/8}$, $f(2) = 2e^{-1/2}$ and $f(4) = 4e^{-2}$ so the absolute minimum is $-e^{-1/8}$ at $x = -1$. and the absolute maximum is $2e^{-1/2}$ at $x = 2$.

(b) On $[0, \infty)$ we have $f(x) \geq 0$ so the absolute minimum is $f(0) = 0$ at $x = 0$. Since $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{e^{x^2/9}} = 0$, the maximum must occur at an interior point of $[0, \infty)$, in particular at a point where $f'(x) = 0$. From part (a) we therefore know that the maximum occurs at $x = 2$ and is $2e^{-1/2}$.

(3) Find the critical numbers and singularities of $h(x) = \begin{cases} x^3 - 6x^2 + 3x & x \leq 3 \\ \sin(2\pi x) - 18 & x \geq 3 \end{cases}$.

Solution: For $x \neq 3$ we have

$$h'(x) = \begin{cases} 3x^2 - 12x + 3 & x < 3 \\ 2\pi \cos(2\pi x) & x > 3 \end{cases}.$$

In particular, the critical points where $x < 3$ are where $3(x^2 - 4x + 1) = 0$, that is where $x = \frac{4\pm\sqrt{12}}{2} = 2 \pm \sqrt{3}$ – but (**pitfall!**)$2 + \sqrt{3} > 2 + 1 = 3$ so the only critical point to the left of 3 is $2 - \sqrt{3}$. To the right of 3 we need to solve $\cos(2\pi x) = 0$ which occurs when $x = \frac{1}{4} + \frac{k}{2}$, $k \in \mathbb{Z}$ – but we need $x > 3$ so the critical points to the right of 3 are $\left\{\frac{1}{4} + \frac{k}{2} \mid k \geq 6\right\}$. At $x = 3$ h is continuous

$$\lim_{x\to 3^-} h(x) = h(3) = 27 - 54 + 9 = -18 = \sin(6\pi) - 18 = \lim_{x\to 3^+} h(x)$$

but singular: by the definition of the derivative, on the left we have

$$\lim_{x\to 3^-} \frac{h(x) - h(3)}{x - 3} = \left(3x^2 - 12x + 3\right) \restriction_{x=3} = -6$$

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while

$$\lim_{x\to 3^+}\frac{h(x)-h(3)}{x-3} = (\cos(2\pi x))\restriction_{x=3} = \cos(6\pi) = 1\,.$$

2. The Mean Value Theorem

(1) Let $f(x) = e^x$ on the interval $[0,1]$. Find all values of c so that $f'(c) = \frac{f(1)-f(0)}{1-0}$.

Solution: $f'(x) = e^x$ so we need to solve

$$e^c = \frac{e-1}{1}$$

and get $\boxed{c = \log(e-1)}$.

(2) Let $f(x) = |x|$ on the interval $[-1,2]$. Find all values of c so that $f'(c) = \frac{f(2)-f(-1)}{2-(-1)}$.

Solution: $f'(x) = \begin{cases} 1 & x>0 \\ -1 & x<0 \end{cases}$ while $\frac{f(2)-f(-1)}{2-(-1)} = \frac{2-1}{2+1} = \frac{1}{3}$ and there are no solutions (but f' is not differentiable on all of $(-1,2)$, so MVT is not violated).

(3) Show that $f(x) = 3x^3+2x-1+\sin x$ has exactly one real zero. (Hint: let a,b be zeroes of f. The MVT will find c such that $f'(c) =$?)

Solution: f is everywhere differentiable (defined by formula) and in particular everywhere continuous.

- At least one zero: We have $f(10) = 3019 + \sin 10 > 3000 > 0$ and $f(-10) = -3021 - \sin(10) < -3000 < 0$ so by the IVT f has a zero in $(-10,10)$.
- Not more than one: Suppose $a<b$ where both zeroes of f,so that $f(a) = f(b) = 0$. Since f is everywhere differentiable there would be $c\in(a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{0-0}{b-a} = 0\,.$$

But $f'(x) = 9x^2+2+\cos x \geq 2+\cos x \geq 2-1 = 1 > 0$ for all x, so f' is nowhere vanishing. This contradicts the existence of c, hence of the distinct zeroes a,b.

(4) Suppose $f(1) = 3$ and $-3 \leq f'(x) \leq 2$ for $x\in[1,4]$. What can you say about $f(4)$?

Solution: Since f is differentiable on $[-1,4]$ it's continuous there and the MVT applies. There is therefore $c\in(1,4)$ such that

$$f'(c) = \frac{f(4)-f(1)}{4-1}\,.$$

We were given that $-3\leq f'(c) \leq 2$ and $f(1)=3$ so

$$-3 \leq \frac{f(4)-3}{3} \leq 2\,.$$

Multiplying by the positive number 3 we get

$$-9 \leq f(4)-3 \leq 6\,.$$

Adding 3 we now get

$$\boxed{-6 \leq f(4) \leq 9\,.}$$

(5) Show that $|\sin a - \sin b| \le |a - b|$ for all a, b.

Solution: When a, b are equal both sides are zero, so suppose $a \neq b$. Since $f(x) = \sin x$ is everywhere differentiable it's also everywhere continuous. We may thus apply the MVT: given $a \neq b$ there is c between a, b such that

$$\frac{\sin a - \sin b}{a - b} = f'(c) = \cos c\,.$$

Taking absolute values we get

$$\left|\frac{\sin a - \sin b}{a - b}\right| = |\cos c| \le 1$$

so

$$\frac{|\sin a - \sin b|}{|a - b|} \le 1\,,$$

and the claim follows upon multiplication by $|a - b|$.

(6) Let $x > 0$. Show that $e^x > 1 + x$ and that $\log(1 + x) \le x$.

Solution: For the first claim, the function $f(x) = e^x$ is everywhere continuous and differentiable. Given $x > 0$ by the MVT there is $c \in (0, x)$ such that

$$\frac{e^x - e^0}{x - 0} = f'(c) = e^c\,.$$

Since $c > 0$ we have $e^c > e^0 = 1$ so

$$\frac{e^x - 1}{x} > 1\,.$$

Multiplying by x and adding 1 we get

$$e^x > x + 1\,.$$

Similarly, let $g(x) = \log(1 + x)$ which is again differentiable and continuous on $x > -1$. In particular applying the MVT on the interval $[0, x]$ we get $c \in (0, x)$ such that

$$\frac{\log(1 + x) - \log(1 + 0)}{x - 0} = \frac{g(x) - g(0)}{x - 0} = g'(c) = \frac{1}{1 + c}\,,$$

where at the end we used $g'(x) = \frac{1}{1+x}$. But if $c > 0$ then $\frac{1}{1+c} < \frac{1}{1} = 1$ and using $\log 1 = 0$ we conclude

$$\frac{\log(1 + x)}{x} < 1$$

and hence

$$\log(1 + x) < x\,.$$