MATH 100 – SOLUTIONS TO WORKSHEET 14
TAYLOR POLYNOMIALS

1. Taylor expansion of e^x

(1) Let $f(x) = e^x$
   (a) Find $f(0), f'(0), f^{(2)}(0), \cdots$
   (b) Find a simple polynomial $T_0(x)$ such that $T_0(0) = f(0)$.
   (c) Find a simple polynomial $T_1(x)$ such that $T_1(0) = f(0)$ and $T_1'(0) = f'(0)$.
   (d) Find a simple polynomial $T_2(x)$ such that $T_2(0) = f(0)$, $T_2'(0) = f'(0)$ and $T_2^{(2)}(0) = f^{(2)}(0)$.
   (e) Find a simple polynomial $T_3(x)$ such that $T_3^{(k)}(0) = f^{(k)}(0)$ for $0 \le k \le 3$.

   Solution:
   (a) $f'(x) = e^x$, $f''(x) = e^x$, and in fact $f^{(k)}(x) = e^x$ for all x. Since $e^0 = 1$ we see that $f^k(0) = 1$ for all k.
   (b) $\boxed{T_0(x) = 1}$ works.
   (c) Suppose $T_1(x) = a + bx$. Then $T_1(0) = a$ so need $a = f(0) = 1$. Also, $T_1'(x) = b$ so need $b = 1$ and get

$$\boxed{T_1(x) = 1 + x\,.}$$

   (d) Suppose $T_2(x) = 1 + x + cx^2$. Then $T_2(0) = 1$, $T_2'(x) = 1 + 2cx$ so $T_2'(0) = 1$. Also, $T_2''(x) = 2c$ so to get $T_2''(0) = f''(0) = 1$ need $c = \frac{1}{2}$ and we get

$$\boxed{T_2(x) = 1 + x + \frac{1}{2}x^2\,.}$$

   (e) Suppose $T_3(x) = 1 + x + \frac{1}{2}x^2 + dx^3$. Then $T_3^{(3)}(x) = 3 \cdot 2d$ so to get $T_2^{(3)}(0) = f^{(3)}(0) = 1$ need $d = \frac{1}{6}$ and we get

$$\boxed{T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\,.}$$

2. Do the same with $f(x) = \ln x$ about $x = 1$.**Solution**:
   (1) $f'(x) = \frac{1}{x}$, $f^{(2)}(x) = -\frac{1}{x^2}$, $f^{(3)}(x) = \frac{2}{x^3}$. Thus $f(1) = 0$, $f^{(1)}(1) = 1$, $f^{(2)}(1) = -1$, $f^{(3)}(1) = 2$.
   (2) $\boxed{T_0(x) = 0}$.
   (3) Suppose $T_1(x) = 0 + bx$. Then we need $b = f'(1) = 1$. so we get

$$\boxed{T_1(x) = x\,.}$$

   (4) Suppose $T_2(x) = x + cx^2$. Then $T_2^{(2)}(x) = 2c$ so to get $T_2^{(2)}(1) = f^{(2)}(1) = -1$ need $c = -\frac{1}{2}$ and we get

$$\boxed{T_2(x) = x - \frac{1}{2}x^2\,.}$$

   (5) Suppose $T_3(x) = x - \frac{1}{2}x^2 + dx^3$. Then $T_3^{(3)}(x) = 3 \cdot 2d$ so to get $T_2^{(3)}(1) = f^{(3)}(1) = 2$ need $d = \frac{2}{6} = 3$ and we get

$$\boxed{T_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3\,.}$$

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2. General formula

The nth order Taylor expansion of $f(x)$ about $x = a$ is the polynomial

$$T_n(x) = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n$$

where $c_k = \frac{f^{(k)}(a)}{k!}$.

(1) Find the 4th order Maclaurin expansion of $\frac{1}{1-x}$.

Solution: For $f(x) = \frac{1}{1-x}$ we have $f^{(1)}(x) = \frac{1}{(1-x)^2}$, $f^{(2)}(x) = \frac{2}{(1-x)^3}$, $f^{(3)}(x) = \frac{2\cdot 3}{(1-x)^4}$, $f^{(4)}(x) = \frac{2\cdot 3\cdot 4}{(1-x)^5}$. We therefore have $f(0) = 1 = 0!$, $f^{(1)}(0) = 1 = 1!$, $f^{(2)}(0) = 2!$, $f^{(3)}(0) = 3!$, $f^{(4)}(0) = 4!$ and hence

$$\begin{aligned} T_4(x) &= \frac{0!}{0!} + \frac{1!}{1!}x + \frac{2!}{2!}x^2 + \frac{3!}{3!}x^3 + \frac{4!}{4!}x^4 \\ &= \boxed{1 + x + x^2 + x^3 + x^4}. \end{aligned}$$

(2) Find the nth order expansion of $\cos x$.

Solution: For $g(x) = \cos x$ the derivatives are $g^{(0)}(x) = \cos x$, $g^{(1)}(x) = -\sin x$, $g^{(2)}(x) = -\cos x$, $g^{(3)}(x) = \sin x$, $g^{(4)}(x) = \cos x$ and then the derivatives repeat. It follows that the derivatives at zero are $1, 0, -1, 1$, and then repeat periodically. We conclude that the odd terms all vanish, and the even terms are the same as those of e^x but switch sign:

$$\boxed{\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \frac{1}{8!}x^8 - \cdots + \frac{(-1)^n}{(2n)!}x^{2n}}.$$

3. New from old

(1) Find the 3rd order Taylor expansion of $\sqrt{x}$ about $x = 4$ and use it to approximate $\sqrt{4.1}$.

Solution: Let $f(x) = \sqrt{x}$, $a = 4$. Then $f'(x) = \frac{1}{2}x^{-1/2}$, $f^{(2)}(x) = -\frac{1}{4}x^{-3/2}$, $f^{(3)}(x) = \frac{3}{8}x^{-5/2}$. Therefore $f(4) = 2$, $f'(4) = \frac{1}{4}$, $f^{(2)}(4) = -\frac{1}{32}$, $f^{(3)}(4) = \frac{3}{256}$. We have

$$\begin{aligned} T_3(x) &= f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 \\ &= 2 + \frac{1}{4}(x-4) + \frac{1}{2}\left(-\frac{1}{32}\right)(x-4)^2 + \frac{1}{6}\left(\frac{3}{256}\right)(x-4)^3 \\ &= 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3. \end{aligned}$$

For us $x = 4.1$ so $x - a = 4.1 - 4 = \frac{1}{10}$ and we get

$$\boxed{\sqrt{4.1} \approx T_3(4.1) = 2 + \frac{1}{40} - \frac{1}{6400} + \frac{1}{512,000}}.$$

(2) Find the 3rd order Taylor expansion of $\sqrt{x} + 3x$ about $x = 4$.

Solution: We already know the expansion of $\sqrt{x}$. For $3x$ the value at 4 is 12 and the slope is 3 so $3x = 12 + 3(x-4)$. Thus

$$\begin{aligned} \sqrt{x} + 3x &\approx \left(2 + \frac{1}{4}(x-4) + \frac{1}{2}\left(-\frac{1}{32}\right)(x-4)^2 + \frac{1}{6}\left(\frac{3}{256}\right)(x-4)^3\right) + (12 + 3(x-4)) \\ &= \boxed{14 + 3\frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3}. \end{aligned}$$

(3) Find the 8th order expansion of $f(x) = e^{x^2} + \cos(2x)$. What is $f^{(6)}(0)$?

Solution: We already know that $e^y \approx 1 + y + \frac{1}{2}y^2 + \frac{1}{6}y^3 + \frac{1}{24}y^4$ to fourth order. Plugging in x^2 we find

$$e^{x^2} \approx 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8$$

(and see that terms y^k with $k > 4$ would give terms x^{2k} with $2k > 8$ so not relevant). Similarly, using $\cos y \approx 1 - \frac{1}{2}y^2 + \frac{1}{24}y^4 - \frac{1}{6!}y^6 + \frac{1}{8!}y^8$ and plugging in $y = 2x$ we get

$$\begin{aligned}\cos(2x) &\approx 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 - \frac{1}{6!}(2x)^6 + \frac{1}{8!}(2x)^8 \\ &= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \frac{2}{215}x^8 .\end{aligned}$$

We conclude that the 8th order expansion is:

$$\begin{aligned}e^{x^2} + \cos(2x) &\approx (1+1) + (1-2)x^2 + (\frac{1}{2} + \frac{2}{3})x^4 + \left(\frac{1}{6} - \frac{4}{45}\right)x^6 + \left(\frac{1}{24} + \frac{2}{315}\right)x^8 \\ &= \boxed{2 - x^2 + \frac{7}{6}x^4 + \frac{7}{90}x^6 + \frac{121}{2520}x^8 .}\end{aligned}$$

We now <u>use the Taylor expansion rule in reverse</u>: the coefficient of x^6 is $\frac{7}{90}$, but it is also $\frac{f^{(6)}(0)}{6!}$ so

$$\frac{f^{(6)}(0)}{6!} = \frac{7}{90}$$

and

$$f^{(6)}(0) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}{2 \cdot 5 \cdot 9} = 56 .$$