MATH 100 – WORKSHEET 9
IMPLICIT DIFFERENTIATION

1. Implicit Differentiation

(1) Find the line tangent to the curve $y^2 = 4x^3 + 2x$ at the point $(2, 6)$.

Solution: We differentiate the equation with respect to x to get using the chain rule:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(y^2) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(4x^3 + 2x\right) \\ 2y\frac{\mathrm{d}y}{\mathrm{d}x} &= 12x^2 + 2 \end{aligned}$$

so that

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x^2 + 1}{y}\,.$$

We conclude that at the point $x = 2, y = 6$ we have $y' = \frac{25}{6}$, so the tangent line has the equation

$$\boxed{y = \frac{25}{6}(x - 2) + 6\,.}$$

(2) Find y'' if $x^5 + y^5 = 10$.

Solution 1: We differentiate the equation with respect to x and get

$$5x^4 + 5y^4 y' = 0$$

so

$$y' = -\frac{x^4}{y^4}\,.$$

We differentiate again and apply the quotient rule and the chain rule to get:

$$y'' = -\frac{4x^3y^4 - x^4 4y^3 y'}{y^8} = -4\frac{x^3 y - x^4 y'}{y^5}\,.$$

Susbtituting our formula for y' we get

$$\boxed{y'' = -4\frac{x^3 y + x^8/y^4}{y^5} = -4\frac{x^3y^5 + x^8}{y^9}\,.}$$

Solution 2: We differentiate the equation with respect to x and get

$$5x^4 + 5y^4 y' = 0\,.$$

We then differentiate again to get

$$20x^3 + 20y^3(y')^2 + 5y^4 y'' = 0\,.$$

Dividing by 5 and substituting $y' = -\frac{x^4}{y^4}$ gives

$$\begin{aligned} -y^4 y'' &= 4x^3 + 4y^3\frac{x^8}{y^8} \\ &= 4\frac{x^3y^5 + x^8}{y^5}\,. \end{aligned}$$

Now solve for y''.

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(3) (Final 2012) Find the slope of the tangent line to the curve $y + x\cos y = \cos x$ at the point $(0, 1)$.
Solution: We differentiate the equation to get

$$y' + \cos y - x \sin y \cdot y' = -\sin x$$

so that

$$y' = -\frac{\cos y + \sin x}{1 - x \sin y}.$$

For $x = 0$, $y = 1$ this reads

$$y' = -\frac{\cos 1 + \sin 0}{1 - 0 \sin 1} = -\cos 1$$

so the tangent line has slope $-\cos 1$.

(4) Find y' if $(x + y)\sin(xy) = x^2$.
Solution: We differentiate the equation to get

$$(1 + y')\sin(xy) + (x + y)\cos(xy)\,(y + xy') = 2x$$

that is

$$y'\left(\sin(xy) + (x + y)\cos(xy)x\right) = 2x - \sin(xy) + (x + y)\cos(xy)y$$

and hence

$$y' = \frac{2x - \sin(xy) + (x + y)\cos(xy)y}{\sin(xy) + (x + y)\cos(xy)x}.$$

2. Inverse trig functions

(1) (Evaluation)
(a) (Final 2014) Find $\arcsin\left(\sin\left(\frac{31\pi}{11}\right)\right)$.
Solution: We need to find θ such that $\sin\theta = \sin\frac{31\pi}{11}$ and such that $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Now $\sin\left(\frac{31\pi}{11}\right) = \sin\left(\frac{31\pi}{11} - 2\pi\right) = \sin\left(\frac{31-22}{11}\pi\right) = \sin\left(\frac{9}{11}\pi\right)$. Also, $\sin(\pi - \alpha) = \sin\alpha$ so

$$\sin\left(\frac{31}{11}\pi\right) = \sin\left(\frac{9}{11}\pi\right) = \sin\left(\pi - \frac{9}{11}\pi\right) = \sin\left(\frac{2}{11}\pi\right).$$

But $\frac{2}{11}\pi$ is in the desired range, so $\boxed{\theta = \frac{2}{11}\pi}$.

(b) Find $\tan(\arccos(0.4))$
Solution: Let $0 \le \theta \le \pi$ be such that $\cos\theta = 0.4$. We need to find $\tan\theta$. First, since $0.4 > 0$, $0 < \theta < \frac{\pi}{2}$ so $\sin\theta > 0$. Second, by Pythagoras $\sin^2\theta + \cos^2\theta = 1$ so $\sin\theta = \sqrt{1 - \cos^2\theta} = \sqrt{0.84}$. We conclude that

$$\tan(\arccos(0.4)) = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{0.84}}{0.4}.$$

(2) Differentiation
(a) Find $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x))$
Solution: By the chain rule $\frac{\mathrm{d}}{\mathrm{d}x}(\arcsin(2x)) = \frac{1}{\sqrt{1-(2x)^2}}\frac{\mathrm{d}}{\mathrm{d}x}(2x) = \frac{2}{\sqrt{1-4x^2}}$.

(b) Find $\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1 + (\arctan(x))^2}$.
Solution: By the chain rule,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\sqrt{1 + (\arctan(x))^2} &= \frac{1}{2\sqrt{1 + (\arctan(x))^2}}\frac{\mathrm{d}}{\mathrm{d}x}\left[1 + (\arctan(x))^2\right] \\
&= \frac{1}{2\sqrt{1 + (\arctan(x))^2}}(2\arctan(x))\frac{\mathrm{d}}{\mathrm{d}x}[\arctan x] \\
&= \frac{\arctan x}{\sqrt{1 + (\arctan(x))^2}} \cdot \frac{1}{1 + x^2}.
\end{aligned}$$

(c) Find y' if $y = \arcsin\left(e^{5x}\right)$. What is the domain of the functions y, y'?
Solution 1: By the chain rule, $y' = \frac{1}{\sqrt{1-(e^{5x})^2}} e^{5x} \cdot 5$ so

$$\boxed{y' = \frac{5e^{5x}}{\sqrt{1-e^{10x}}}.}$$

The function e^{5x} is defined everywhere, but the domain of arcsin is $[-1, 1]$ so the domain of y is $\left\{x \mid -1 \leq e^{5x} \leq 1\right\}$. But $e^{5x} > 0$ always, so the domain is $\left\{x \mid e^{5x} \leq 1\right\}$ which is exactly $(-\infty, 0]$. The derivative is defined where y is, except when $e^{10x} = 1$ that is except when $x = 0$ and its domain is $(-\infty, 0)$.