MATH 100 – WORKSHEET 8
INVERSE FUNCTIONS

1. More chain rule

(1) Differentiate
  (a) $7x + \cos(x^n)$
    Solution: $\frac{\mathrm{d}}{\mathrm{d}x}\left[7x + \cos(x^n)\right] \overset{\text{sum}}{=} \frac{\mathrm{d}}{\mathrm{d}x}(7x) + \frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(x^n)\right] \overset{\text{chain}}{=} 7 - \sin(x^n)\frac{\mathrm{d}}{\mathrm{d}x}\left[x^n\right] \overset{\text{power}}{=} 7 - n\sin(x^n)x^{n-1}$.
  (b) (Final 2012) $e^{(\sin x)^2}$
    Solution: $\frac{\mathrm{d}}{\mathrm{d}x}\left[e^{(\sin x)^2}\right] \overset{\text{chain}}{=} e^{(\sin x)^2}\frac{\mathrm{d}}{\mathrm{d}x}\left[(\sin x)^2\right] \overset{\text{chain}}{=} e^{(\sin x)^2}2(\sin x)\frac{\mathrm{d}}{\mathrm{d}x}\left[\sin x\right] = -2e^{(\sin x)^2}\sin x\cos x$.

(2) Is there c such that the function is differentiable for all $x > -1$?

$$f(x) = \begin{cases} \frac{\cos(x^2)}{x+1} & x \le 0 \\ cx + x^2 + 1 & x > 0 \end{cases}$$

Solution 1: For $a \neq 0$ f is defined by a well-behaved formula near a, so $f'(a)$ exists (the denominator in the first part only vanishes at $x = -1$). For $f'(0)$ We need to evaluate the limit $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$. Note that $f(0) = \frac{\cos 0}{0+1} = 1 = c0 + 0^2 + 1$ no matter what c is. We now compute separately from the left and the right:

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{\left(ch + h^2 + 1\right) - 1}{h} = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\left(cx + x^2 + 1\right) = \left[c + 2x\right]_{x=0} = c$$

and

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{\frac{\cos(h^2)}{h+1} - 1}{h} = \left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\left(\frac{\cos(x^2)}{x+1}\right) = \left[\frac{-2x\sin(x^2)(x+1) - \cos(x^2)\cdot 1}{(x+1)^2}\right]_{x=0} = -1\,.$$

It follows that $\lim_{h\to 0}\frac{f(h)-f(0)}{h}$ exists if and only if $\boxed{c = -1}$.

Solution 2: We first check that f is continuous at $x = 0$. Indeed $\lim_{h\to 0^-} f(x) = f(0) = 1$ since $\frac{\cos(x^2)}{x+1}$ is continuous at $x = 0$ (defined by well-defined formula). Also, $\lim_{h\to 0^+} f(x) = \lim_{h\to 0^+}\left(cx + x^2 + 1\right) = c0 + 0^2 + 1 = 1$. We now differentiate both formulas and check that the derivatives match. (justification needed:) Once we know f is continuous this is enough because the derivatives are themselves limits from the right and the left. We have (need to do the computation as in solution 1 above)

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\left(cx + x^2 + 1\right) = c$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\left(\frac{\cos(x^2)}{x+1}\right) = -1$$

so the function is differentiable iff $c = -1$.

Remark 2: DO NOT be confused by how we check for continuity and take $\lim_{x\to 0} f'(x)$.

2. Inverse Functions

(1) Find the function inverse to $y = x^7 + 3$.

Solution: We solve for x to get: $x^7 = y - 3$ so $x = (y-3)^{1/7}$ and then switch x, y to get

$$\boxed{y = (x-3)^{1/7}}$$

(can also switch x, y first and then solve).

(2) Consider the function $y = \sqrt{x-1}$ on $x \ge 1$.

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(a) Find the inverse function, in the form $x = g(y)$.

Solution: Solving for x we find $y^2 = x - 1$ so $\boxed{x = y^2 + 1}$.

(b) Find $\frac{dy}{dx}$, $\frac{dx}{dy}$ and calculate their product.

Solution: $\boxed{\frac{dy}{dx} = \frac{1}{2\sqrt{x-1}}}$ while $\boxed{\frac{dx}{dy} = \frac{d}{dy}\left(y^2+1\right) = 2y}$ so $\frac{dy}{dx} \cdot \frac{dx}{dy} = \frac{2y}{2\sqrt{x-1}} = 1$ since $y = \sqrt{x-1}$.

(3) Does $y = x^2$ have an inverse?

Solution: No, because on its full domain it takes most values twice. For example, $1^2 = -1^2$.