Math 322: Problem Set 10 (due 20/11/2014)

P1. Find a group G and three pairwise disjoint subgroups A,B,C such that the multiplication map $A\times B\times C\to G$ is not injective.

DEFINITION. Let G be a group. Call $g\in G$ a *torsion element* if g has finite order ($g^k=e$ for some $k\neq 0$), and write G_{tors} for the set of torsion elements. Say that g is *p-power torsion* if its order is a power of p. For an abelian group write $A\left[p^\infty\right]$ for the set of its p-power torsion elements.

P2. (Torsion) Let G,H be groups, A an abelian group.
- (a) If G is finite then $G=G_{\mathrm{tors}}$. Give an example of an infinite consisting entirely of torsion elements.
- (b) Show that $f(G_{\mathrm{tors}})\subset H_{\mathrm{tors}}$ for any $f\in\mathrm{Hom}(G,H)$.
- (c) $A_{\mathrm{tors}}=\bigcup_{n\geq 1}A[n]$, $A\left[p^\infty\right]=\bigcup_{r=0}^{\infty}A\left[p^r\right]$.
- (d) Let $X\in\mathrm{GL}_n(\mathbb{R})$ be a torsion element. Show that the eigenvalues of X are (possibly complex) roots of unity.
- (e) Find $X,Y\in\mathrm{GL}_n(\mathbb{R})_{\mathrm{tors}}$ such that XY has infinite order.

Abelian groups

1. (First do problem P2) Fix an abelian group A.
   - (a) Show that A_{tors} and $A\left[p^\infty\right]$ are subgroups of A.
   - (b) Show that $A\left[p^\infty\right]$ is the p-Sylow subgroup of A.
   - — It follows that, if A is finite, $A=\prod_p A\left[p^\infty\right]$ as an internal direct product.
   - (c) Show that A/A_{tors} is *torsion-free*: $(A/A_{\mathrm{tors}})_{\mathrm{tors}}=\{e\}$.

2. Find the Sylow subgroups of $C_{360}\times C_{300}\times C_{200}\times C_{150}$.

Nilpotent groups and torsion

3. Let G be *two-step nilpotent*, in that $G/Z(G)$ is abelian.
   PRAC Verify that the Heisenberg group (PS7 problem P2) is two-step nilpotent.
   - (a) For $x,y\in G$ let $[x,y]=xyx^{-1}y^{-1}$ be their commutator. Show that $[x,y]\in Z(G)$ for all G (hint: this is purely formal).
   - (b) Let $x,y\in G$ and $z,z'\in Z(G)$. Show that $[x,y]=[xz,yz']$ and conclude that the commutator induces a map $G/Z\times G/Z\to Z$.
   - (c) Show that this map is *anti-symmetric*: $[\bar{y},\bar{x}]=[\bar{x},\bar{y}]^{-1}$ and *biadditive:* $[\bar{x}\bar{x}',\bar{y}]=[\bar{x},\bar{y}]\,[\bar{x}',\bar{y}]$, $[\bar{x},\bar{y}\bar{y}']=[\bar{x},\bar{y}]\,[\bar{x},\bar{y}']$.

   RMK In fact, a two-step nilpotent group is more-or-less determined by the abelian groups $A=G/Z(G)$, $Z=Z(G)$ and the anti-symmetric biadditive form$[\cdot,\cdot]:A\times A\to Z$.

4. (Torsion in nilpotent groups) Continue with the hypotheses of problem 3.
   - (a) Let $x,y\in G$ and suppose that $x\in G_{\mathrm{tors}}$. Show that $[x,y]\in Z(G)_{\mathrm{tors}}$.
   - (*b) (The hard part). Show that G_{tors} is a subgroup of G.

RMK In general, a group is 0-step nilpotent if it is trivial, $(k+1)$-step nilpotent if $G/Z(G)$ is k-step nilpotent, and *nilpotent* if it is k-step nilpotent for some k. A variant on the argument above shows that the set of torsion elements of any nilpotent group is a subgroup.