## Math 322: Problem Set 5 (due 9/10/2014)

### Practice problems

P1. $H = \{\mathrm{id}, (12)\}$ and $K = \{\mathrm{id}, (123), (132)\}$ are two subgroups of $S_3$. Compute the coset spaces $S_3/H$, $H\backslash S_3$, $S_3/K$, $K\backslash S_3$.

P2. Let $K < H < G$ be groups with $G$ finite. Use Lagrange's Theorem to show $[G:K] = [G:H]\,[H:K]$.

P3. Let $N < G$ satisfy for all $g \in G$ that $gNg^{-1} \subset N$. Show that for all $g \in G$, $gNg^{-1} = N$.

P4. Let $N < G$ satisfy for all $g_1, g_2 \in G$ that if $g_1 \equiv_L g_1' \,(N)$ and $g_2 \equiv_L g_2' \,(N)$ then $g_1 g_2 \equiv_L g_1' g_2' \,(N)$.

(a) Show that for any $g \in G$, $n \in G$ we have $gng^{-1} \equiv_L e \,(N)$, and conclude that $gNg^{-1} = N$.

(b) Give $G/\equiv_L (N)$ a group structure, and construct a homomorphism $q\colon G \to G/N$ such that $N = \mathrm{Ker}(q)$. Conclude that $N$ is normal.

### Cosets, normal subgroups and quotients

1. (Normalizers and centralizers) Let $G$ be a group, $X \subset G$ a subset. The *centralizer* of $X$ (in $G$) is $Z_G(X) = \{g \in G \mid \forall x \in X : gx = xg\}$ (in particular $Z(G) = Z_G(G)$ is called the *centre* of $G$). The *normalizer* of $X$ (in $G$) is $N_G(X) = \left\{g \in G \mid gXg^{-1} = X\right\}$. Fix $H < G$.

(a) Show that $N_G(X) < G$.

PRAC Show that $Z_G(X) < N_G(X)$.

(b) Show $H < N_G(H)$.

PRAC Let $H < K < G$. Show that $H \lhd K$ iff $K \subset N_G(H)$. In particular, $H \lhd G$ iff $N_G(H) = G$.

(c) Show that $Z(G)$ is a normal, abelian subgroup of $G$.

PRAC Show that $H \cap Z_G(H) = Z(H)$, in particular that $H \subset Z_G(H)$ iff $H$ is abelian.

2. (Semidirect products) Let $H, K < G$ and consider the map $f\colon H \times K \to G$ given by $f(h,k) = hk$. Recall that the image of this map is denoted $HK$.

(a) Show that $f$ is injective iff $H \cap K = \{e\}$.

SUPP For $x \in HK$ give a bijection $f^{-1}(x) \leftrightarrow H \cap K$, hence a bijection $H \times K \leftrightarrow HK \times H \cap K$.

PRAC Show $H < N_G(K) \iff \forall h \in H : hKh^{-1} = K$. In this case we say "$H$ normalizes $K$".

(b) Suppose $H$ normalizes $K$. Show that $HK$ is a subgroup of $G$ and that $\langle H \cup K \rangle = HK$. Show that $K \lhd HK$ (hint: you need to show that $HK < N_G(K)$ and already know that $H, K$ separately are contained there).

DEF If $H < N_G(K)$ and $H \cap K = \{e\}$ we call $HK$ the (*internal*) *semidirect product* of $H$ and $K$. We write $HK = H \ltimes K$ (combining the symbols for product and normal subgroup).

(c) Let $HK$ be the semidirect product of $H, K$ and let $q\colon HK \to (HK)/K$ be the quotient map. Directly show that the restriction $q\restriction_H\colon H \to (HK)/K$ is an isomorphism. (Hint: what is the kernel? what is the image?)

PRAC Let $g, h \in G$. Show that $gh = hg$ iff the *commutator* $[g,h] = ghg^{-1}h^{-1}$ has $[g,h] = e$.

— For parts (c),(d) suppose that $H, K$ normalize each other and that $H \cap K = \{e\}$

(d) Show that $H, K$ *commute*: $hk = kh$ whenever $h \in H$, $k \in K$.

(e) Show that the map $f$ is an isomorphism onto its image (it's a bijection by part (a); you need to show it is a group homomorphism).

DEF In that case we say $HK$ is the (*internal*) *direct product* of $H$ and $K$.

PRAC Let $G = \mathrm{GL}_2(\mathbb{R})$ be the group of $2 \times 2$ invertible matrices. We will consider the subgroups $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G \middle| ad \neq 0 \right\}$, $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G \middle| ad \neq 0 \right\}$ and $N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}$.

(a) Show that these really are subgroups. Evidently $N, A \subset B \subset G$.

(b) Show that $A \simeq (\mathbb{R}^{\times})^2 = \mathbb{R}^{\times} \times \mathbb{R}^{\times}$. Show that $N \simeq \mathbb{R}^{+}$.

(b) Show that $B = N \rtimes A$(you need to show that $B = NA$, that $A \cap N = \{I\}$, and that $N \triangleleft B$).

(c) Directly show that for any fixed $a, d$ with $ad \neq 0$ we have $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} N = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid b \in \mathbb{R} \right\}$, demonstrating part of 2(c).

3. Let $K < H < G$ be a chain of subgroups. Let $R \subset G$ be a system of representatives for $G/H$ and let $S \subset H$ be a system of representatives for $H/K$.

(a) Show that the map $R \times S \to RS$ given by $(r, s) \mapsto rs$ is a bijection.

(b) Show that $RS = \{rs \mid r \in R, s \in S\}$ is a system of representatives for $G/K$, and conclude that $[G : K] = [G : H]\,[H : K]$.

RMK See P1 for a numerical proof in the finite case.

4. In a previous problem set we defined the subgroup $P_n = \{\sigma \in S_n \mid \sigma(n) = n\}$ of $S_n$. We now give an explicit description of $S_n/P_n$ and use that to inductively determine the order of $S_n$.

(a) Show that for $\tau, \tau' \in S_n$ we have $\tau P_n = \tau' P_n$ iff $\tau(n) = \tau'(n)$, and conclude that $[S_n : P_n] = n$.

(b) Show that $P_n \simeq S_{n-1}$.

(c) Combine (a),(b) into a proof by induction that $|S_n| = n!$.

**Challenge problems**

5. Let $G$ be a group

(a) Suppose that $x^2 = e$ for all $x \in G$. Show that $G$ is abelian.

(**b) Suppose that $G$ has $n$ elements, at least $\frac{3}{4}n$ of which have order 2. Then $G$ is abelian.

6**. Let $G$ be group of order $n$. Show that there is $X \subset G$ of size at most $\log_2 n$ such that $G = \langle X \rangle$.

## Supplementary Problems: Quotients and the abelianization

A. (The universal property of $G/N$) Let $N \lhd G$. An "abstract quotient" of a group $G$ is a group $\bar{G}$, together with a homomorphism $\bar{q} \colon G \to \bar{G}$ such that the property for any $f \colon G \to H$ with kernel containing $N$ there is a unique $\bar{f} \colon \bar{G} \to H$ with $f = \bar{f} \circ \bar{q}$ (in class we saw that the quotient group $G/N$ has this property). Suppose that $\left(\bar{G}', \bar{q}'\right)$ is another abstract quotient. Show that there is a uinque isomorphism $\varphi \colon \bar{G} \to \bar{G}'$ such that $\bar{q}' = \varphi \circ \bar{q}$.

B. (The Correspondence Theorem) Let $f \in \mathrm{Hom}(G,H)$, and let $K = \mathrm{Ker}\, f$.
   (a) For every subgroup $M$, $K < M < G$, show that $f(M)$ is a subgroup of the image $f(G)$.
   (b) Show that the map $M \mapsto f(M)$ is a bijection between the set of subgroups of $G$ containing $K$ and the set of subgroups of the image $f(G)$.
   (c) Show that this bijection preserves inclusion of subgroups, and also index and normality (in $G$ and $f(G)$, respectively)
   (d) Let $N \lhd G$ and let $X \subset G$ be such that its image in $G/N$ generate $G/N$. Show that $N \cup X$ generate $G$.

D. (The derived subgroup and abelian quotients) Fix a group $G$ and recall that notation $[g,h] = ghg^{-1}h^{-1}$.
   (a) Let $f \in \mathrm{Hom}(G,H)$ be a homomorphism. Show that $f([g,h]) = [f(g), f(h)]$ for all $g,h \in G$.
   (b) Deduce from (a) that the set of commutators is invariant under conjugation.
   DEF For $H,K < G$ set $[H,K] = \langle \{[h,k] \mid h,k \in G\} \rangle$ – note that this is the *subgroup* generated by those commutators, not just the set of commutators. In particular, we write $G' = [G,G]$ for the *derived subgroup* (or *commutator subgroup*) of $G$, the subgroup generated by all the commutators.
   (c) Show that $G'$ is normal in $G$.
   (d) Show that $G^{\mathrm{ab}} \stackrel{\mathrm{def}}{=} G/G'$ is abelian (hint: apply (a) to the quotient map).
   DEF we call $G^{\mathrm{ab}}$ the *abelianization* of $G$.
   (e) Let $N \lhd G$. Show that $G/N$ is abelian iff $G' \subset N$.
   (f) Let $A$ be an abelian group and let $q \colon G \to G^{\mathrm{ab}}$ be the quotient map. Show that the map $\Phi \colon \mathrm{Hom}(G^{\mathrm{ab}}, A) \to \mathrm{Hom}(G,A)$ given by $\Phi(f) = f \circ q$ is a bijection.

E. Compute the derived subgroup and the abelianization of the groups: $C_n, D_{2n}, S_n, \mathrm{GL}_n(\mathbb{R})$.