MATH 100 – WORKSHEET 10
LOGARITHMIC DIFFERENTIATION, APPLICATIONS

1. Logarithmic Differentiation

$$\boxed{(\log x)' = \frac{1}{x}} \qquad\qquad \boxed{f' = f \times (\log f)'}$$

(1) Differentiate.

(a) $\frac{x^5 \cos x}{\sqrt{5+x}}$

Solution: $\log \frac{x^5 \cos x}{\sqrt{5+x}} = \log(x^5) + \log(\cos x) - \log \sqrt{5+x} = 5 \log x + \log \cos x - \frac{1}{2} \log(5+x)$. Thus

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^5 \cos x}{\sqrt{5+x}}\right) &= \left(\frac{x^5 \cos x}{\sqrt{5+x}}\right) \cdot \frac{\mathrm{d}}{\mathrm{d}x}\left(5 \log x + \log \cos x - \frac{1}{2} \log(5+x)\right) \\ &= \left(\frac{x^5 \cos x}{\sqrt{5+x}}\right)\left(\frac{5}{x} - \frac{\sin x}{\cos x} - \frac{1}{2(5+x)}\right). \end{aligned}$$

(b) x^x

Solution: Applying logarithmic differentiation, and $\log a^b = b \log a$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^x) &\overset{\text{log diff}}{=} x^x \frac{\mathrm{d}}{\mathrm{d}x}(\log(x^x)) = x^x \frac{\mathrm{d}}{\mathrm{d}x}(x \log x) \\ &\overset{\text{pdt rule}}{=} x^x \left(\log x + \frac{x}{x}\right) = (1 + \log x)\, x^x . \end{aligned}$$

(c) $(\log x)^{\cos x}$

Solution: Applying logarithmic differentiation, and $\log a^b = b \log a$, we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left((\log x)^{\cos x}\right) &\overset{\text{log diff}}{=} \left((\log x)^{\cos x}\right) \frac{\mathrm{d}}{\mathrm{d}x}\left(\log\left((\log x)^{\cos x}\right)\right) \\ &= (\log x)^{\cos x} \frac{\mathrm{d}}{\mathrm{d}x}\left((\cos x) \cdot \log \log x\right) \\ &\overset{\text{pdt rule}}{=} (\log x)^{\cos x}\left(-\sin x \log \log x + \cos x \frac{\mathrm{d}}{\mathrm{d}x} \log \log x\right) \\ &\overset{\text{chain rule}}{=} (\log x)^{\cos x}\left(-\sin x \log \log x + \cos x \frac{1}{\log x} \frac{1}{x}\right) \\ &= (\log x)^{\cos x}\left(\frac{\cos x}{x \log x} - \sin x \log \log x\right). \end{aligned}$$

Date: 9/10/2014.

2. Applications

Object moves by $s = f(t)$. Then the *velocity* is $v(t) = \frac{\mathrm{d}s}{\mathrm{d}t}$ and the *acceleration* is $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2}$

(1) The position of a particle at time t is given by $f(t) = \frac{1}{\pi}\sin(\pi t)$.
  (a) Find the velocity at time t, and specifically at $t = 3$.
  Solution: $v(t) = \frac{\mathrm{d}s}{\mathrm{d}t} = \cos(\pi t)$ by the chain rule. At $t = 3$ this reads $v(3) = \cos(3\pi) = -1$.
  (b) When is the particle accelerating? Decelerating?
  Solution: $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t} = -\pi\sin(\pi t)$ by the chain rule. This is negative for $0 < t < \pi$, positive for $\pi < t < 2\pi$, etc.

(2)
  (a) Water is filling a cylindrical container of radius $r = 10$cm. Suppose that at time t the height of the water is $\left(t + t^2\right)$ cm. How fast is the volume growing?
  Solution: The volume of a cylinder of radius r and height h is $V = \pi r^2 h$. In centimetres we are given $r = 10$, $h(t) = t + t^2$so

$$V(t) = 100\pi\left(t + t^2\right)$$

  and

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 100\pi\,(1 + 2t)\,\frac{1}{\text{unit time}}\,.$$

  (b) A rocket is flying in space. The momentum of the rocket is given by the formula $p = mv$, where m is the mass and v is the velocity. At a time where the mass of the rocket is $m = 1000$kg and its velocity is $v = 500\frac{\mathrm{m}}{\mathrm{s}}$ the rocket is accelerating at the rate $a = 20\frac{\mathrm{m}}{\mathrm{s}^2}$ and losing mass at the rate $10\frac{\mathrm{kg}}{\mathrm{s}}$. Find the rate of change of the momentum with time.
  Solution: Appling the product rule and using $a = \frac{\mathrm{d}v}{\mathrm{d}t}$ we have:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}v + m\frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}v + ma\,.$$

  We now plug in the given values $\frac{\mathrm{d}m}{\mathrm{d}t} = -10\frac{\mathrm{kg}}{\mathrm{s}}$ (the rocket is *ejecting* mass), $v = 500\frac{\mathrm{m}}{\mathrm{s}}$, $m = 1000$kg and $a = 20\frac{\mathrm{m}}{\mathrm{s}^2}$ to get at the given instant:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = -10\cdot 500 + 1000\cdot 20 = 15{,}000\mathrm{N}\,.$$

(3) A ball is falling from rest in air. Its height at time t is given by

$$h(t) = H_0 - gt_0\left(t + t_0 e^{-t/t_0} - t_0\right)$$

where H_0 is the initial height and t_0 is a constant.
  (a) Find the velocity of the ball. $v(t) = \frac{\mathrm{d}h}{\mathrm{d}t}(t) = 0 - gt_0\left(1 + t_0\left(-\frac{1}{t_0}\right)e^{-t/t_0} - 0\right) = -gt_0\left(1 - e^{-t/t_0}\right)$.
  (b) Find the acceleration. $a(t) = \frac{\mathrm{d}v}{\mathrm{d}t}(t) = -gt_0\left(0 - \left(-\frac{1}{t_0}\right)e^{-t/t_0}\right) = -ge^{-t/t_0}$.
  (c) Find $\lim_{t\to\infty} v(t) = \lim_{t\to\infty}(-gt_0)\left(1 - e^{-t/t_0}\right) = -gt_0\left(1 - \lim_{t\to\infty} e^{-t/t_0}\right)$. Now $e^{-t/t_0} = \frac{1}{e^{t/t_0}} \xrightarrow[t\to\infty]{} 0$ since e^{t/t_0} grows without bound. We get that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -gt_0\,.$$