MATH 253 – WORKSHEET 29
TRIPLE INTEGRALS

(1) Evaluate $\iiint_E e^{x+y+z}\,\mathrm{d}V$ where E is the tetrahedron with vertices $(3,0,0)$, $(0,2,0)$, $(0,0,1)$, $(0,0,0)$.

Solution: The tetrahedron can be though of as the pyramid with its base the triangle in the xy plane given by $(3,0,0)$, $(0,2,0)$, $(0,0,0)$ and its apex the point $(0,0,1)$. Thus (x,y) will range in the triangle, and z will range from zero to the value of z on the plane through the points $(3,0,0)$, $(0,2,0)$, $(0,0,1)$. The equation of that plane is $\frac{x}{3}+\frac{y}{2}+\frac{z}{1}=1$ (the coefficients of x,y,z chosen so that the plane passes through the given points). The line connecting $(3,0,0)$and $(0,2,0)$ in the xy plane is the intersection of this plane with the plane $z=0$ so has the equation $\frac{x}{3}+\frac{y}{2}=1$. Our integral is therefore

$$\begin{aligned}
\int_{x=0}^{x=3}\mathrm{d}x\int_{y=0}^{y=2\left(1-\frac{x}{3}\right)}\mathrm{d}y\int_{z=0}^{z=1-\frac{x}{3}-\frac{y}{2}}\mathrm{d}z e^{x+y+z} &= \int_{x=0}^{x=3}\mathrm{d}x e^x\int_{y=0}^{y=2\left(1-\frac{x}{3}\right)}\mathrm{d}y e^y\int_{z=0}^{z=1-\frac{x}{3}-\frac{y}{2}}\mathrm{d}z e^z \\
&= \int_{x=0}^{x=3}\mathrm{d}x e^x\int_{y=0}^{y=2\left(1-\frac{x}{3}\right)}\mathrm{d}y e^y\left[e^{1-\frac{x}{3}-\frac{y}{2}}-1\right] \\
&= \int_{x=0}^{x=3}\mathrm{d}x e^x\int_{y=0}^{y=2\left(1-\frac{x}{3}\right)}\mathrm{d}y\left(e^{1-\frac{x}{3}}e^{\frac{6}{2}}-e^y\right) \\
&= \int_{x=0}^{x=3}\mathrm{d}x e^x\left[2e^{1-\frac{x}{3}}e^{\frac{y}{2}}-e^y\right]_{y=0}^{y=2\left(1-\frac{x}{3}\right)} \\
&= \int_{x=0}^{x=3}\mathrm{d}x e^x\left(2e^{1-\frac{x}{3}}e^{1-\frac{x}{3}}-e^{2\left(1-\frac{x}{3}\right)}-2e^{1-\frac{x}{3}}+1\right) \\
&= \int_{x=0}^{x=3}\mathrm{d}x e^x\left(e^{2\left(1-\frac{x}{3}\right)}-2e^{1-\frac{x}{3}}+1\right) \\
&= \int_{x=0}^{x=3}\mathrm{d}x\left(e^{2+\frac{x}{3}}-2e^{1+\frac{2x}{3}}+e^x\right) \\
&= \left[3e^{2+\frac{x}{3}}-3e^{1+\frac{2x}{3}}+e^x\right]_{x=0}^{x=3} \\
&= 3e^3-3e^3+e^3-3e^2+3e-1 \\
&= e^3-3e^2+3e-1=(e-1)^3\,.
\end{aligned}$$

Exercise. Do the integral in differenent orders, for example $\int_{z=0}^{z=1}\mathrm{d}z\int_{x=0}^{x=3(1-z)}\mathrm{d}x\int_{y=0}^{y=2\left(1-z-\frac{x}{3}\right)}\mathrm{d}y e^{x+y+z}$.

(2) Let E be the solid region between the plane $x=4$ and the paraboloid $x=y^2+z^2$. Set up the limits for the integral $\iiint_E f\,\mathrm{d}V$

(a) Integrating $\int\mathrm{d}y\int\mathrm{d}z\int\mathrm{d}x f$.

Solution: The region is a solid of revolution: revolve the function $x=y^2$ about the x-axis. It has the shape of a bullet. Slices parallel to the yz plane have the shape of discs of radius $\sqrt{x}$ around the x-axis, so the shadow on the xy plane is the disc $x^2+y^2\leq 4$. Given y,z the x range starts from a point on the paraboloid and ends at the "cap" $x=4$ at the base of the bullet. The integral is therefore

$$\iint_{y^2+z^2\leq 4}\mathrm{d}y\,\mathrm{d}z\int_{x=y^2+z^2}^{x=4}\mathrm{d}x f=\int_{y=-2}^{y=+2}\mathrm{d}y\int_{z=-\sqrt{4-z^2}}^{z=+\sqrt{4-z^2}}\mathrm{d}z\int_{x=y^2+z^2}^{x=4}\mathrm{d}x f\,.$$

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(b) Integrating $\int \mathrm{d}x \int \mathrm{d}y \int \mathrm{d}z f$.

Solution: We now use the observation from before directly: the slice parallel to the yz plane at x is the disc $y^2 + z^2 \leq x$. The integral is therefore

$$\int_{x=0}^{x=4} \mathrm{d}x \iint_{y^2+z^2\leq x} \mathrm{d}y\,\mathrm{d}z f = \int_{x=0}^{x=4} \mathrm{d}x \int_{y=-\sqrt{x}}^{y=+\sqrt{x}} \mathrm{d}y \int_{z=-\sqrt{x-z^2}}^{z=+\sqrt{x-z^2}} \mathrm{d}z f\,.$$

(3) Consider the iterated integral $\int_{x=0}^{x=1} \mathrm{d}x \int_{y=\sqrt{x}}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z f$. Write the other 5 equivalent integrals coming from changing the order of integration.

Solution: We note that the range in the inner integral depends on the x, y chosen before but not on the order in which they were chosen, so we can switch the first two variables ignoring the third. Similarly, fixing x leaves us with a two-variable integral in which we can switch order as usual. To start with we switch $\mathrm{d}x\,\mathrm{d}y$ to $\mathrm{d}y\,\mathrm{d}x$. The domain $\{0 \leq x \leq 1,\ \sqrt{x} \leq y \leq 1\}$ is the region above the graph of $y = \sqrt{x}$ and below $y = 1$, but it is also the region between the y-axis and the graph of $x = y^2$, so if

$$I_1 = \int_{x=0}^{x=1} \mathrm{d}x \int_{y=\sqrt{x}}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z f$$

we have

$$I_2 = \int_{y=0}^{y=1} \mathrm{d}y \int_{x=0}^{x=y^2} \mathrm{d}x \int_{z=0}^{z=1-y} \mathrm{d}z f\,.$$

The inner inegral $\mathrm{d}y\,\mathrm{d}z$ in I_1 is the integral over the triangle in the yz plane with vertices $(\sqrt{x}, 0)\,, (1, 0)$ on the y-axis and $(\sqrt{x}, 1-\sqrt{x})$ off it (so the line $z = 1-y$ passes through this point and $(1, 0)$). The z range is thus $[0, 1-\sqrt{x}]$ and the then the y-range is from the line $y = \sqrt{x}$ to the line $y = 1-z$ so the integral also equals

$$I_3 = \int_{x=0}^{x=1} \mathrm{d}x \int_{z=0}^{z=1-\sqrt{x}} \mathrm{d}z \int_{y=\sqrt{x}}^{y=1-z} \mathrm{d}y f\,.$$

We can exchange the first two variables here: the integral is on the triangular wedge lying below the graph of $z = 1 - \sqrt{x}$ in the positive quadrant of the xz plane, which is also the wedge to the left of $x = (1-z)^2$ in the same quadrant. The integral is then

$$I_4 = \int_{z=0}^{z=1} \mathrm{d}z \int_{x=0}^{x=(1-z)^2} \mathrm{d}x \int_{y=\sqrt{x}}^{y=1-z} \mathrm{d}y f\,.$$

Finally, we can exchange the order of the inner integral in I_2 and I_4. In I_2 the inner integral is on the rectangle $[0, y^2] \times [0, 1-z]$ in the xz plane (note the bounds only depend on y!) so can immediately switch the order to get

$$I_5 = \int_{y=0}^{y=1} \mathrm{d}y \int_{z=0}^{z=1-y} \mathrm{d}z \int_{x=0}^{x=y^2} \mathrm{d}x f\,.$$

In I_4 the inner two integrals are in the region of the xy plane, above the graph of $y = \sqrt{x}$ and below the constant line $y = 1-z$ (they meet at $\left((1-z)^2, (1-z)\right)$) (and to the right of the x-axis). Switching the order y will range from 0 to $1-z$ and the horizonal lines (slices) will begin at at the y-axis and end on the graph of $y = \sqrt{x}$, so

$$I_6 = \int_{z=0}^{z=1} \mathrm{d}z \int_{y=0}^{y=1-z} \mathrm{d}y \int_{x=0}^{x=y^2} \mathrm{d}x f\,.$$